Lecture 3

Summary

1. Newtunian determinacy

\[ \frac{d^2 r_i}{dt^2} = \vec{F}_i(t, \vec{r}_i, \vec{v}_i, \vec{u}_i) \]

\((2^{nd} \text{ order equation})\)

2. Inertial mass - separates properties of particles from forces

\[ m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_i, \vec{v}_i, \vec{u}_i) \]

\((\text{right side the same for all particles})\)

3. Galilean invariance - isolated system

\[ \vec{F}_i \rightarrow \vec{F}_i = \vec{F}_i + v_i \vec{u} + \vec{F}_0 \quad ; \quad t \rightarrow t' = t + c \]

\[ \vec{r}_i = \vec{r}_i(\vec{r}_0-\vec{r}_i) \quad \vec{v}_i-\vec{v}_i \quad \vec{u}_i-\vec{u}_i \quad \vec{u}_0-\vec{u}_0 \]

4. Non inertial coordinates systems have "forces" that are proportional to inertial mass

5. Special relativity - changes the relation between inertial coordinate systems - Newton's 2nd law is valid in the particle's instantaneous rest frame

Conservative forces

A set of forces \( \vec{F}_i(\vec{r}, \vec{v}, \vec{u}) \) is called conservative if
If \( \bar{F}_i = -\nabla \cdot V(\bar{r}_i, \bar{r}_r) \) where \( V(\bar{r}_i, \bar{r}_r) \) is a single valued potential function.

Note: \( N \) vector forces can be expressed in terms of a single scalar function.

Let \( \bar{y}_i(u) \) be paths from \( \bar{r}_{i0} \) to \( \bar{r}_{i0}' \):
\[
\bar{y}_i(0) = \bar{r}_{iA} \quad \bar{y}_i(1) = \bar{r}_{iB}
\]

The work done in moving \( N \) particles against these forces along the paths \( \bar{y}_i(u) \) is
\[
W_{AB} = -\sum_{i=1}^{N} \int_{\bar{r}_{iA}}^{\bar{r}_{iB}} \bar{F}_i(\hat{\bar{y}}_i(u), \bar{r}_r(u)) \cdot \frac{d\bar{y}_i}{du} \, du
\]

If the forces are conservative, this become:
\[
W_{AB} = -\sum_{i=1}^{N} \int_{\bar{r}_{iA}}^{\bar{r}_{iB}} (-\nabla \cdot \bar{F}_i(\bar{y}_i, \bar{r}_r)) \cdot \frac{d\bar{y}_i}{du} \, du = \]
\[ = \int_0^1 \frac{d}{d\lambda} V(\tilde{\gamma}_i(\lambda) - \tilde{\gamma}_{\nu}(\lambda)) \, d\lambda = \]
\[ = V(\tilde{\gamma}_i(1) - \tilde{\gamma}_{\nu}(1)) - V(\tilde{\gamma}_i(0) - \tilde{\gamma}_{\nu}(0)) = \]
\[ = V(\tilde{\gamma}_{\mu 0} - \tilde{\gamma}_{\nu 0}) - V(\tilde{\gamma}_{\mu 1} - \tilde{\gamma}_{\nu 1}) \]

This shows that the work done against a conservative force is independent of path - it only depends on the endpoint.

For an isolated system obeying Galilean relativity, the potential is a function of coordinate differences

\[ V(\tilde{r}_2 - \tilde{r}_1, \ldots, \tilde{r}_n - \tilde{r}_1) \]

In this case

\[ \sum_{i=1}^{2} \tilde{F}_i = - \sum_{i=1}^{2} \nabla_{\tilde{r}_i} V = \]
\[ = - \sum_{j=2}^{n} \nabla_{\tilde{r}_j} V \cdot \frac{d\tilde{r}_j}{d\tilde{r}_1} - \sum_{j=1}^{n} \nabla_{\tilde{r}_j} V \]
\[ = \sum_{j=2}^{n} (-\nabla_{\tilde{r}_j} V) \nabla_{\tilde{r}_1} V = 0 \]
This means that the net force on an isolated system of particles obeying Galilean relativity is 0.

For 2 particles, this means
\[ \vec{F}_{12} = -\vec{F}_{21} \]

which is Newton's third law.

\* Laws

1. In the absence of forces, free particles (in inertial coordinate systems) move with constant velocity.
2. \[ M_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i \]
3. \[ \vec{F}_{ii} = -\vec{F}_{ii} \]

Conservation laws for systems of particles from Galilean invariance
consider a system of $N$ particles

* assume that the forces on each of the particles due to the other $N-1$ particles is given by a Galilean invariant potential

* there are also external forces $\mathbf{F}_i^{ext}$ on each of the particles

translational invariance and momentum conservation

define $\mathbf{p}_i = m_i \mathbf{v}_i$; momentum of $i^{th}$ particle
$M = \sum m_i$; mass of system
$\mathbf{P} = \sum \mathbf{p}_i$; momentum of system

$$\frac{d}{dt} \mathbf{P} = \sum_{i=1}^N \frac{d\mathbf{p}_i}{dt} = \sum \mathbf{F}_i = \sum (-\nabla V_i(r_i, r_c) + \mathbf{F}_i^{ext})$$

vanishes for Galilean invariance

$$\sum_{i \neq j} F_{ij} = 0$$

The total momentum of the system is conserved if the sum of all of the external forces on the system add to zero.
Energy conservation (time independent force)

\[ T = \sum \frac{\vec{p}_i^2}{2m_i} \quad V = \text{potential} = V(\vec{r}_i, \vec{p}_i) \]

\[
\frac{d}{dt} (T+V) = \sum_{i=1}^{n} \left( 2p_i \cdot \frac{1}{2m_i} \frac{dp_i}{dt} + \nabla_i V \cdot \frac{dp_i}{dt} \right)
\]

From the second law \( \frac{dp_i}{dt} = F_i = -\nabla_i V \), \( \frac{\vec{p}_i}{m_i} = \vec{V}_i \)

\[
\frac{d}{dt} (T+V) = \sum (\vec{V}_i \cdot (-\nabla_i V) + (\nabla_i V) \cdot \vec{V}_i) = 0
\]

This shows that the sum of the total kinetic + potential energy of the system is conserved (again this assumes no external force)

\[
\frac{d}{dt} (T+V) = 0
\]

Next define

\[ \vec{l}_i = \vec{r}_i \times \vec{p}_i \quad \text{angular momentum of particle } i \text{ about origin} \]

\[ \vec{L} = \sum \vec{l}_i \quad \text{total angular momentum of system} \]
\[
\frac{d\vec{L}}{dt} = \sum \frac{d\vec{P}_i}{dt} \times \vec{P}_i + \sum \vec{r}_i \times \frac{d\vec{P}_i}{dt}
\]
\[
= \sum \frac{1}{m_i} \vec{p}_i \times \vec{v}_i + \sum \vec{r}_i \times \vec{F}_i
\]
\[
\text{The part is called the torque}
\]

For the case where all of the forces are internal,
\[
\sum \vec{F}_i \times \vec{F}_i = -\sum \vec{F}_i \times \vec{v}_i \vec{V}
\]
\[
= -\sum \vec{F}_i \times \vec{F}_{ij} = -\sum \frac{1}{2} (\vec{F}_i \times \vec{F}_{ij} - \vec{F}_j \times \vec{F}_{ji})
\]
\[
= -\frac{1}{2} (\vec{F}_i - \vec{F}_j) \times \vec{F}_{ij}
\]

If the force between particle $i$ and $j$ is such that $\vec{F}_{ij} = (\vec{F}_i - \vec{F}_j) \times (\vec{r}_i - \vec{r}_j)$, this vanishes when we take the cross product.

In the absence of external torques, the total angular momentum is conserved.
If we define
\[
\vec{R} = \frac{\vec{p}}{m} = \frac{\sum p_i}{\sum m_i} = \frac{d}{dt} \left( \sum \frac{m_i \vec{r}_i}{m} \right)
\]
the \( \vec{R} = \sum \frac{m_i \vec{r}_i}{m} \)
is called the center of mass.

In the absence of external forces the center of mass of a many particle system moves like a free particle.

This justifies our assumption that a small system could be treated as a point particle.

Lagrangian equations and constraints

While Newton's second law describes the motion of particles given a potential in a set of forces, in many cases the forces are not known.
as an example - consider a simple pendulum

\[ \begin{align*}
\text{In this case} \\
(1) & \text{ while the motion is in the } xy \text{ plane - the } x \text{ and } y \\
& \text{ coordinates are not independent} \\
& x^2 + y^2 = L^2 \\
& \text{ where } L \text{ is the length of the pendulum} \\
(2) & \text{ the tension force is unknown} \\
(3) & \text{ the tension force does no work on the system because it is always directed } T \\
& \text{ to the motion} 
\end{align*} \]
even this simple system presents new challenges

The condition $x^2 + y^2 = L^2$ is called a holonomic constraint. In general a holonomic constraint is a relation among coordinates of the form

$$h_i(\vec{r}_1, \ldots, \vec{r}_n) = 0$$

For a given system there can be several holonomic constraints.

To treat systems with holonomic constraints, start with Newton's 2nd Law

$$\frac{d\vec{p}_i}{dt} - \vec{F}_i = 0$$

* use the holonomic constraint to write

$$\sum_j \vec{V}_j h(\vec{r}_j, \vec{r}_i) \cdot \delta \vec{F}_i = 0$$
For small displacements, $\delta \vec{r}_i$, consistent with the constraints, no work is done

$$\left( \frac{d\vec{p}_i}{dt} - \vec{F}_i \right) \cdot \delta \vec{r}_i = 0$$

$$\vec{F}_i = \vec{F}_i^c + \vec{F}_i^a$$

where $\vec{F}_i^c =$ constraint force

$\vec{F}_i^a =$ applied force

In this case $\sum_i \vec{F}_i^c \cdot \delta \vec{r}_i = 0$ because the forces of constraint do no work, then the equations above become

$$\left( \frac{d\vec{p}_i}{dt} - \vec{F}_i^a \right) \cdot \delta \vec{r}_i = 0$$

where all of the constraint forces have been eliminated.

When the constraints are holonomic, we can use them to eliminate unnecessary coordinates and express all 8N coordinates in terms of a set of independent generalized coordinates.
$F_1 = F_1(q_1, \ldots, q_{1c})$

$\vdots$

$F_N = F_n(q_1, q_{1c})$

Example

$x^2 + y^2 = L^2 = 0$

$x = L \cos \theta$

$y = L \sin \theta \quad \{ x \text{ generalized coordinate} \}$

$x = \pm \sqrt{L^2 - y^2} \quad \{ y \text{ generalized coordinate} \}$

$y = y$

$x = \pm \sqrt{L^2 - y^2} \quad \{ y \text{ generalized coordinate} \}$

In general the holonomic constraint could be time dependent

$x^2 + y^2 = L^2(t)$

To construct displacements consistent with the constraints

$\delta \tilde{F}_i = \int \frac{\partial \tilde{F}_i}{\partial q_e} \delta q_e$

$x \tilde{F}_i(q_1, q_{1c})$ uses all of the constraints
using this we get
\[ 0 = \sum_i \left( \frac{d\vec{p}_i}{dt} - \vec{F}_i^A \right) \cdot \delta \vec{r}_i = \sum_i \left( \frac{d\vec{p}_i}{dt} - \vec{F}_i^A \right) \sum_j \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \delta \vec{q}_j \]

because the displacements \( \delta \vec{q}_j \) are independent we get \( k \) equations
\[ 0 = \sum_i \left( \frac{d\vec{p}_i}{dt} - \vec{F}_i^A \right) \cdot \frac{\partial \vec{r}_i}{\partial \vec{q}_k} = 0 \quad \forall k = 1, \ldots, k \]

the next step is to express these equations in terms of the generalized coordinates.

Step 1
\[ \frac{d\vec{p}_i}{dt} = \frac{d}{dt} \left( m_i \frac{d\vec{r}_i}{dt} \right) \cdot \frac{\partial \vec{r}_i}{\partial \vec{q}_k} = \frac{d}{dt} \left( m_i \frac{d\vec{r}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \right) - m_i \frac{d\vec{r}_i}{dt} \frac{d}{dt} \left( \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \right) \]

To calculate the second term note
\[ \frac{d}{dt} \left( \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \right) = \sum_j \frac{\partial^2 \vec{r}_i}{\partial \vec{q}_j \partial \vec{q}_k} \frac{d\vec{q}_j}{dt} + \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \frac{d}{dt} = 2 \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \left( \frac{\partial \vec{r}_i}{\partial \vec{q}_k} \frac{d\vec{q}_j}{dt} + \frac{\vec{r}_i}{\partial \vec{q}_j} \frac{d\vec{q}_j}{dt} \right) \]

\[ \frac{d\vec{r}_i}{dt} = \sum_j \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \frac{d\vec{q}_j}{dt} + \frac{\vec{r}_i}{\partial \vec{q}_j} \frac{d\vec{q}_j}{dt} = D \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \frac{d\vec{q}_j}{dt} + \frac{\vec{r}_i}{\partial \vec{q}_j} \frac{d\vec{q}_j}{dt} \]
This gives
\[ 0 = \sum_{i=1}^{n} \left( \frac{d\vec{p}_i}{dt} - F_i^A \right) \cdot \frac{\partial F_i}{\partial q_k} = \]
\[ = \sum_{i=1}^{n} \left\{ \frac{d}{dt} \left( m_i \frac{\dot{\vec{r}}_i}{\partial q_k} \cdot \frac{\partial F_i}{\partial q_k} \right) - m_i \frac{\dot{\vec{r}}_i}{\partial q_k} \frac{d}{dt} \left( \frac{\partial F_i}{\partial q_k} \right) - F_i^A \frac{\partial F_i}{\partial q_k} \right\} = \]
\[ = \sum_{i=1}^{n} \left\{ \frac{d}{dt} \left( \frac{\partial F_i}{\partial q_k} \right) \left( \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \right) - m_i \ddot{\vec{r}}_i \cdot \frac{\partial F_i}{\partial q_k} - F_i^A \frac{\partial F_i}{\partial q_k} \right\} = \]
\[ = \sum_{i=1}^{n} \left( \frac{d}{dt} \left( \frac{\partial F_i}{\partial q_k} \right) - \frac{\partial F_i}{\partial q_k} \right) \left( \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \right) - F_i^A \frac{\partial F_i}{\partial q_k} = 0 \]

This gives
\[ \frac{d}{dt} \left( \frac{\partial F_i}{\partial q_k} \right) - \frac{\partial F_i}{\partial q_k} = Q_k \quad Q_k = 2 \sum_i F_i^A \frac{\partial F_i}{\partial q_k} \quad k = 1 \cdots K \]

\( Q_k \) is called the generalized force.

For a conservative force
\[ Q_k = \sum_i F_i^A \cdot \frac{\partial F_i}{\partial q_k} = -\sum_i \vec{v}_i \cdot \frac{\partial F_i}{\partial q_k} = -\frac{\partial V}{\partial q_k} \]
\[ = \frac{d}{dt} \left( \frac{2V}{2q_n} \right) - \frac{2V}{2q_n} \]
\[ (\text{here } \frac{2V}{2q_n} = 0) \]
This gives

$$\left( \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} \right) (T-V) = 0$$

The quantity

$$\mathcal{L} = T-V = \mathcal{L}(\dot{q}_i, \dot{q}_k, q_i, q_k, \epsilon)$$

is called the Lagrangian of the system. The equations

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

are called Lagrange's equations.

Solving them gives

$$q_i(t)$$

$$\vec{r}_i(t) = \vec{r}_i(q_i(t), ..., q_n(t), t)$$

- Note the Lagrangian is a single scalar function that includes the effect of all forces.
- The forces of constraint no longer appear in the problem.
example: harmonic oscillator

\[ x = L \sin \theta \]
\[ y = L \cos \theta \]
\[ \dot{x} = L \cos \theta \]
\[ \dot{y} = -L \sin \theta \]
\[ v = -mg \]
\[ L = mL^2 \cos \theta \]

\[ \tau = \frac{1}{2} m(x^2 + y^2) = \frac{1}{2} mL^2 \dot{\theta}^2 \]

\[ L = \frac{1}{2} mL^2 \dot{\theta}^2 + mgL \cos \theta \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]

\[ mL^2 \ddot{\theta} + mgl \sin \theta = 0 \]

\[ mL^2 \dddot{\theta} = -mgL \sin \theta \]

\[ \frac{d}{dt} \left( \frac{1}{2} mL^2 \dot{\theta}^2 \right) = \frac{d}{dt} (mgL \cos \theta) \]

\[ \frac{1}{2} mL^2 \ddot{\theta}^2 - mgL \cos \theta = \text{const} \]

\[ \dot{\theta}^2 = \frac{2}{mL^2} \left( \text{const} - mgL \cos \theta \right) \]

\[ \frac{d\theta}{dt} = \sqrt{\frac{2}{mL^2}} \left( mgL \cos \theta - mgL \cos \theta \right) \]

\[ t - t_i = \int_{\theta_i}^{\theta} \frac{d\theta}{\sqrt{\frac{mgL}{2} (\cos \theta - \cos \theta_i)}} \]

Note the tension does not appear.