Lecture 8

Last time

Investigate

$$S^2 A \left[ t_f, t_f \times s_f \right] = 0$$

for

$$\frac{d}{dt} \left( \frac{2L}{S^2 \dot{q}_i} \right) - \frac{2L}{S^2 q_i} = 0$$

subject to the constraint

$$l = \sum_{i} S q_i(t_f) \dot{q}_{i}(t) \quad s q_i(t_r) = s q_i(t_f) = 0$$

(1) $$S^2 A =$$

$$\sum_{i=1}^{n} \int \left( A_{ij}(t) S q_i(t) S q_j(t) + 2 B_{ij}(t) S q_i(t) S q_j(t) \right) dt$$

$$+ C_{ij}(t) S q_i(t) S q_j(t)$$

where

$$A_{ij}(t) = \frac{\partial^2 L}{\partial q_i \partial q_j} (\ddot{q}(t)) = A_{ji}(t)$$

$$B_{ij}(t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} (\ddot{q}(t))$$

$$C_{ij}(t) = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} (\ddot{q}(t)) = C_{ji}(t)$$

and $$\ddot{q}(t)$$ is the solution to Lagrange's equation.
If $\bar{s}_q(t)$ is a local minimum (or maximum) then it is a stationary point of $\bar{s}^2 A(t) + 9 \bar{s}_q^2$.

Because scaling the solution by a constant $\bar{s}_q \to \rho \bar{s}_q$ will change the value of $\bar{s}^2 A$, for any $\bar{s}_q$, this freedom is eliminated by the normalization constraint

$$1 = \int_{\mathbb{T}} \bar{s}^2 \bar{s}_q^2(t) \, dt$$

$$\Rightarrow$$

$$0 = 2 \sum_j \left( -\frac{d}{dt} \left( C_{ij}(t) \bar{s}_q^2(t) + B_{ij}(t) \bar{s}_q(t) \right) + B_{ji}(t) \bar{s}_q^2(t) + A_{ij}(t) \bar{s}_q(t) \right)$$

$$= 2 \lambda \bar{s}_q^2(t)$$

This gives the 2nd order boundary value problem

$$\frac{d}{dt} \left( \sum_{j=1}^{\infty} C_{ij}(t) \bar{s}_q^2(t) + \sum_{j=1}^{\infty} B_{ij}(t) \bar{s}_q(t) \right)$$

$$- \sum_j \left( A_{ij}(t) + B_{ij}(t) \frac{d}{dt} \right) \bar{s}_q(t) = -2 \bar{s}_q(t)$$
To use the constraint multiply by $sg_i(t)$, sum over $i$ and integrate from $t_1 \to t_2$

$$\sum_{ij} \int_{t_1}^{t_2} \left[ s g_i(t) \frac{d}{dt} \left( C_{ij}(t) \frac{d}{dt} s g_j(t) \right) + s g_i(t) \frac{d}{dt} \left( B_{ij}(t) s g_j(t) \right) \right] = -\lambda$$

Integrating the $i(t) = 2$ term by parts, using $sg_i(t_1) = s g_i(t_2)$ gives

$$-\sum_{ij} \int \left[ C_{ij}(t) s g_i(t) s g_j(t) + s g_i(t) B_{ij}(t) s g_j(t) \right] = -\lambda$$

This can be rewritten as

$$\sum_{ij} \int \left[ C_{ij}(t) s g_i(t) s g_j(t) + 2 B_{ij}(t) s g_i(t) s g_j(t) \right] = \lambda$$

which shows that $\lambda$ is the value of the functions when $\bar{sg}$ is the properly normalized eigenfunction with eigenvalue $\lambda$. 
consider

\[ 2 \int \left( q_i(t) \frac{\partial}{\partial t} C_{ij} \frac{d}{dt} q_j + j_i \frac{d}{dt} (B_{ij} q_j) \right. \]

\[ \left. - A_{ij} f_i q_j - B_{ij} f_i \frac{d}{dt} q_j \right) \]

integrating all of the \( \frac{d}{dt} \) by parts

\[ 2 \int \left( q_i(t) \frac{d}{dt} C_{ij} \frac{d}{dt} f_i - q_j B_{ij} \frac{d}{dt} f_i \right. \]

\[ \left. - q_j A_{ij} f_i + q_j \frac{d}{dt} (B_{ij} f_i) \right) \]

using the symmetry of \( A_{ii}, C_{ii} \)

and interchanging \( ij \)

\[ = 2 \int \left\{ q_i \left( \frac{d}{dt} C_{ij} \frac{d}{dt} f_i + \frac{d}{dt} (B_{ij} f_i) \right) \right. \]

\[ \left. - q_i (A_{ij} f_i + B_{ij} \frac{d}{dt} f_i) \right\} \]

which has the same form as the equation at the top of the page with \( q \rightarrow q \)

\[ (f, D g) = (D s, q) \]

This means that the differential operator \( D \) is Hermitian

(Here everything is real)
This is the differential equation form of a real symmetric matrix properties.

\[ (ds^* D s) = \lambda (df^* f) = (s^* D s) \]

Complex conjugation:

\[ (s^* D s^*) = \lambda^* (ff^*) \quad (s^* s^*) = (s^* f) \]

\[ (f, g) = \int_{t_1}^{t_2} s f \cdot (s f) g \cdot (f) d \tau \]

Comparing \( \lambda^* = \lambda \) eigenvalues are real:

\[ D s = \lambda f \quad D (s^* s^*) = \lambda (s^* s^*) \]
\[ D s^* = \lambda f^* \quad D (s^* s^*) = \lambda (s^* s^*) \]

This gives 2 independent real functions with eigenvalues \( \lambda \)

(even could be 0)

\[ \text{Eigenvectors and eigenvalues are real} \]

\[ \lambda (g, f) = (g D s) = (f D g) = \lambda (s, f) \]

\[ (\lambda s - \lambda f \cdot g \cdot f) = 0 \]

either \( \lambda s = \lambda f \) or \( (g, f) = 0 \)
properties

1. \# of eigenvalues and eigenfunctions
2. all eigenfunctions and eigenvalues are real
3. eigenfunctions with different eigenvalues are \perp
4. eigenvalues are discrete and accumulate at \( \infty \)

\[ \text{if } f(t) \text{ satisfies } \int_{t_0}^t |f(t)|^2 \, dt = 1 \]
then
\[ \overline{\tilde{c}_n} = \sum C_n \tilde{q}_n(t) \quad \Sigma |C_n|^2 = 1 \]

Let \( \tilde{q}_n \) \( \lambda_n \) be the \( n \)-th eigenvalue and eigenfunction

\[ \int \left( \tilde{q}_n^T (\tilde{q}^T \tilde{q} + \tilde{q}^T \tilde{B} \tilde{q} + \tilde{q}^T \tilde{A} \tilde{q}) + \tilde{q}^T \tilde{A} \tilde{q} \right) = \lambda \int \tilde{q}^T \tilde{q} \]

\[ \Rightarrow \int \tilde{q}^T \tilde{q} = \tilde{q}^T \tilde{q} \]

\[ \Rightarrow \int \tilde{q}^T \tilde{q} = \tilde{q}^T \tilde{q} \]

\[ \int \tilde{q}_n^T \tilde{q}_m = \lambda_m (\tilde{q}_n^T \tilde{q}_m) = \delta_{nm}, \lambda_m \]
what this means is that
\[ \tilde{q} = \frac{2}{n} c_n s_{q_n} \]
A[\tilde{t}, \tilde{x}, \tilde{q}, \tilde{s}] = \sum c_m c_n \int s_{q_n} s_{q_m} = \sum k^2 \lambda_n
so the value of A[\tilde{t}, \tilde{x}, \tilde{q}, \tilde{s}] for \( \sum s_{q_n} s_{q_n} = 1 \) is given by a weighted average of the eigenvalues that make the second variation stationary.

If all \( \lambda_n > 0 \) then \( \tilde{q}(x) \) is a local minimum.

To understand what happens it is simple to consider the one dimensional case

\[ L = \frac{1}{2} m \dot{x}^2 - V(x) \]
for small time

\[ x(t) = x(t_0) + \dot{x}(t_0)(t-t_0) - \frac{1}{2} m \frac{dV}{dx}(x(t_0)) (t-t_0)^2 + \]
so in very short times the time dependence is due to the initial velocity independent of the potential.
For short time
\[ L \to T = \frac{1}{2} m \dot{x}^2 \]

In this case
\[ c = \frac{d^2 L}{dx^2} = m \]
\[ b = \frac{d^2 L}{dx \, dx} = 0 \]
\[ \lambda = \frac{d^2 L}{dx^2} = 0 \]

The eigenvalue equation is
\[ -\frac{d}{dt} c \frac{d}{dx} \ddot{x} = \lambda \ddot{x} \]
\[ \frac{d^2}{dt^2} m \ddot{x} = -\lambda \ddot{x} \]

The solution that vanishes at 0 or T is
\[ A \sin \left( \frac{n \pi t}{T} \right) \]

Normalization:
\[ 1 = A^2 \int_0^T \sin^2 \left( \frac{n \pi t}{T} \right) dt = A^2 \frac{T}{2} \]
\[ A = \sqrt{\frac{2}{T}} \]
\[ \ddot{x}_n(t) = \sqrt{\frac{2}{T}} \sin \left( \frac{n \pi t}{T} \right) \]
\[ \lambda_n = m \cdot \frac{n^2 \pi^2}{T^2} > 0 \quad n = 1, 2, 3 \ldots \]
In this case we see explicitly that all of the small time eigenvalues are positive.

This means that the stationary points of \( L \) for small times are local minima.

While we did this for one degree of freedom in general the positivity of the kinetic energy leads to positive eigenvalues.

As \( t \) is increased from \( t_0 \), at some point the forces become important, if we think of the eigenfunction being parameterized as functions of \( t_0 \), they will move - some may cross a zero and become negative. Then the solution of Lagrange's equation will remain stationary - but not be a local minimum of the action.
in general - a solution \( \vec{s} \) with 0 eigenvalue will be a solution of

\[
- \frac{\varepsilon}{2} \frac{d}{dt} \left( (C_{ij} \vec{s}) \vec{s} + (B_{ij} \vec{s}) \right) + \frac{2}{2} (A_{ij} \vec{s} \vec{q} + B_{ij} \vec{s} \vec{q}) = 0
\]

This is called the Jacobi equation. While it has solutions, they do not necessarily satisfy the boundary conditions.

To understand this better consider the 1 degree of freedom case again

\[
L = \frac{1}{2} m \dot{x}^2 - V(x)
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m \ddot{x} + \frac{\partial V}{\partial x} = 0 \quad \text{solution}
\]

\[
c_1 = m, \quad \Lambda(t) = -\frac{\partial V}{\partial x^2} (x(t))
\]

\[
(- m \frac{d^2}{dt^2} + \Lambda(t)) \delta x = 0 \quad \text{(Jacobi Eq.)}
\]

Let \( \Delta q(t, p) \) be a solution of Lagrange's equations with

1. \( \Delta q(t_0) = 0, \quad \frac{d}{dt} \Delta q(t_0) = \frac{p}{m} \)
In this case we consider different solutions of Lagrange's equations with the same starting point, but different initial velocities $X(v_0, t)$.

For each $v$ this satisfies

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

If we take the $v$ derivative note

$$(1) \quad \frac{d}{dv} x(v_0, t) = 0$$

This is because all curves in different $v$ all start at $x(t_0) = x(t_0)$ (fixed)

$$(2) \quad \frac{d}{dv} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} \right)$$

$$= \frac{\partial^2 L}{\partial x^2} \frac{d^2 x}{dV} + \frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{d \dot{x}}{dv} - \frac{\partial^2 L}{\partial \dot{x}^2} \frac{d \dot{x}}{dv} - \frac{\partial L}{\partial x} \frac{d x}{dv}$$

$$= \frac{\partial}{\partial x} \left( C \frac{d}{dt} \left( \frac{dx}{dv} \right) + B \frac{dx}{dv} \right) - A \left( \frac{dx}{dv} \right) - B \frac{d}{dv} \left( \frac{dx}{dv} \right) = 0$$

So we see

$$\frac{d}{dt} \left( t, v_0, t_0 \right) = J(t, v)$$
This satisfies the Jacobi equation, this solution does not have to vanish at $t = t_f$

If it vanishes

1. $t_f$ is called a conjugate point.
2. $\frac{dx}{dt}(t_f, t_i, v) = 0$
3. This means that at $t = t_f$ the solutions have the same coordinate, independent of $v_i$.

So we see at conjugate points there are many solutions of Lagrange's equations that have the same initial and final points.

The analog of Fermat principle is a focus of many rays that start at the same points.
Fields

Consider a string of length $L$ along the $x$ axis with mass $M$ tension $T$ and fixed endpoints.

Let the amplitude of the string at $x, t$ be $y(x, t)$.

To treat this as a mechanical system,

1. Break the string into $N$ parts of mass
   \[ \Delta m = \frac{M}{N} \]
   and length
   \[ \Delta L = \frac{L}{N} \]

   Let $y_n$ be the coordinate of the $n$ segment.

   \[
   \sqrt{y_n - y_{n-1}}
   \]

   For small $\Delta$ the force on $y_n$ is approximately

   \[ -T \left( \frac{y_n - y_{n-1}}{\Delta L} - \frac{y_{n+1} - y_n}{\Delta L} \right) \Delta L \]
Newton's Law gives

\[
\frac{\mu}{N} \frac{d^2 y}{dt^2} = -T \frac{d^2 y}{dx^2} \quad \text{(1)}
\]

\[
\frac{\mu \partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0
\]

This is called the wave equation.

Consider

\[
A = \int \left( \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T \left( \frac{\partial y}{\partial x} \right)^2 \right) \, dx \, dt
\]

Let \[y = y_0 + \delta y, \quad \delta y(0t) = \delta y(Lt) = 0\]

\[\delta A = \int \left( \frac{1}{2} \mu \frac{\partial \delta y}{\partial t} \cdot \delta y - \frac{1}{2} T \frac{\partial \delta y}{\partial x} \cdot \delta y \right)
\]

\[\int \left( -\mu \frac{\partial^2 y}{\partial t^2} \cdot \delta y + T \frac{\partial^2 y}{\partial x^2} \delta y \right)
\]

This requires \[\delta y(x_0) = \delta y(x_f) = 0 \quad \text{all time}\]

and \[\delta y'(x_0) = \delta y'(x_f) = 0 \quad \text{all } x\]

We recover equation of motion assuming this means that we fix the endpoints and the initial and final shape of the string.
The Lagrangian approach to fields normally starts from a Lagrangian, and then uses the principle of stationary action to derive equations of motion for the field.