Lecture 43_20
Final Exam Review

• Point particles, coordinates, velocity, acceleration.

• Newtonian principle of determinancy. Motion described by a second order differential equation.

• Inertial coordinate systems and inertial mass. Leads to Newton’s first and second laws:

\[
\mathbf{F} = m \frac{d^2 \mathbf{r}}{dt^2}
\]

Separates particle properties from particle-independent properties.

• Principle of Galilean relativity - determines form of forces in inertial coordinate systems. Rotationally covariant, independent of time, and depends on coordinate and velocity differences.

• Galilean group - relates different inertial coordinate systems. Generated by space and time translations, rotations and shifts of origin by constant velocity.

• Newton’s second law in non-inertial coordinate systems → mass dependent inertial forces

\[
m_i \frac{d^2 y^k_i}{dt^2} = \sum \frac{\partial y^k_i}{\partial r^l_i} F^l_i +
m_i \sum \frac{\partial^2 y^k_i}{\partial r^l_i \partial r^m_i} \frac{\partial r^l_i}{\partial y^s_i} \frac{\partial y^m_i}{\partial y^s_i} \left( \frac{dy^s_i}{dt} - \frac{\partial y^s_i}{\partial t} \right) \left( \frac{dy^l_i}{dt} - \frac{\partial y^l_i}{\partial t} \right) + 2m_i \frac{\partial^2 y^k_i}{\partial r^l_i \partial t} \frac{\partial r^l_i}{\partial y^s_i} \left( \frac{dy^s_i}{dt} - \frac{\partial y^s_i}{\partial t} \right) + m_i \frac{\partial^2 y^k_i}{\partial t^2}
\]
• Local solutions - convert to coupled integral equations. Iterating converges for short times and nice forces.

• Systems of interacting particles.

• Conservative forces, potentials, Newton’s third law.

• Center of mass, total momentum, total angular momentum. In the absence of external forces center of mass behave like a free particle. Justifies using point particles.

• Conservation of energy (conservative forces), conservation of linear momentum (no external forces), conservation of angular momentum (no external torques)

• Holonomic constraints, generalized coordinates

\[ h_i(\mathbf{r}_1, \cdots, \mathbf{r}_{Nt}) = 0 \]

\[ \mathbf{r}_i = \mathbf{r}_i(q_1, \cdots q_k) \]

• D’Alembert’s principle - constraint forces do not work.

• Generalized force

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) = Q_i = \sum \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_i} \]

\( T \) kinetic energy, \( Q_i \) generalized force

• Lagrange’s equations - conservative forces

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0
\]

\( L = T - V \)
Eliminates forces of constraint, flexible choice of generalized coordinates.

- Non-conservative forces - friction and electromagnetic forces.

\[ Q_\gamma = \frac{\partial P}{\partial \dot{q}_i} \quad P = -\sum_j \frac{\alpha_j}{n} (\dot{r}_j \cdot \dot{r}_j)^{n/2} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \]

- Finding forces of constraint - method of Lagrange multipliers

\[ L \rightarrow L + \sum_k \lambda_k h_k \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} - \sum_j \lambda_j \frac{\partial h_k}{\partial q_i} = 0 \]

\[ Q_i = \sum_j \lambda_j \frac{\partial h_k}{\partial q_i} \]

- Variational calculus, first variation, best linear approximation at a point

\[ \delta F[\gamma_0, \delta \gamma] = \frac{d}{d\lambda} F[\gamma_0 + \lambda \delta \gamma]_{\lambda=0} \]

- Stationary points \( \delta F[\gamma_0, \delta \gamma] = 0 \) for all \( \delta \gamma \). Necessary conditions for minima and maxima of functionals.

- Hamilton’s principle = principle of stationary action. Action functional

\[ A[\gamma, t_1, t_2] = \int_{t_1}^{t_2} L(\dot{\gamma}, \gamma, t) \]
\[ \delta A = 0 \] gives Lagrange’s equations as a boundary value problem.

- differential constraints - can replace holonomic constraints - more general than holonomic constraints

\[ 0 = \sum_i f_i(q) \delta q_i \]

- Choice of “time” variable can have consequences.

- Lagrangian’s the differ by a total time derivative:

\[ \delta A = 0 \iff \delta A' = 0 \]

- Second variation, Jacobi equation, caustics - boundary points were boundary value problem does not does not have unique solutions - solution of Lagrange’s equation minimize the action only for small times, stationary otherwise.

\[ \delta^2 F[\gamma_0, \delta \gamma] = \frac{d^2}{d\lambda^2} F[\gamma_0 + \lambda \delta \gamma]_{\lambda=0} \]

- Differential equation for stationary points of second variation is a Hermitian operator; eigenvalues are values of second variation at stationary the points.

- Noether’s theorem - symmetries of the action results in conservation laws

\[ t \rightarrow t' = t + \epsilon \delta t(t) q_i(t) \rightarrow q_i'(t') + \epsilon \delta q_i(q) \]

\[ C = (L - \sum \frac{\partial L}{\partial \dot{q}_i}) \delta t(t) + \sum \frac{\partial L}{\partial q_i} \delta q_i(q) \]

\[ \frac{dC}{dt} = 0 \]

- Applied to get conservation of energy and linear and angular momentum for Galilean invariant action.
• Lagrange’s equations of fields
• Noether’s theorem for fields - result is a conserved current - which leads to conserved quantity

\[ \frac{\partial J^0}{\partial t} + \nabla \cdot J = 0 \]

\[ \frac{d}{dt} \int J^0(r, t) dr = 0 \]

• Stable equilibrium
• Small oscillations about stable equilibrium

\[ T_{ij} := \frac{1}{2} \sum_k \frac{\partial r_k}{\partial q_i} \cdot \frac{\partial r_k}{\partial q_j}(q_0) \quad V_{ij} := \frac{\partial^2 V}{\partial q_i \partial q_j}(q_0) \]

• Normal modes and normal mode frequencies

\[ L \approx \frac{1}{2} (T_{ij} \delta \dot{q}_i \delta \dot{q}_j - V_{ij} \delta q_i \delta q_j) \]

\[ \sum_j (-\omega^2 T_{ij} + V_{ij}) \delta q_j(0) = 0 \]

• Orthogonality conditions

\[ \delta q_i T \delta q_j = \delta_{ij} \quad \delta q_i V \delta q_j = \omega_i^2 \delta_{ij} \]

• Driven oscillators with dissipation - resonance condition

\[ \sum (T_{ij} \delta q''_j + A_{ij} \delta q'_j + V_{ij} \delta q_j) = F_i \]

• Parametric oscillator - unstable points can become stable - solutions at \( t + T \) related to solutions at \( t \) by a sympletic map.

\[ r(t) = M^{t/T} r_p(t) \]
• Rigid body motion - definition of a rigid body
\[ |r_i - r_j| = d_{ij} = \text{constant} \]

• Body fixed and inertial coordinate system related by translations and orthogonal transformations
\[ r_i = r_{0I} + O^t r_{ib} \quad OO^t = I \]

• Equations of motion of a rigid body - rotational and translational motion decouple if the origin of the body fixed system is its center of mass. Motion is pure rotational if origin of body fixed system is fixed in inertial coordinate system.

• Equations of motion in a rotating coordinate system.

• Generalized coordinates - Euler angles - Lagrangian in terms of Euler angles - Inertia tensor

• Cayley Klein parameters ((SU(2)) - form of Lagrangian in Cayley Klein parameters.

• Inertia tensor, principal axes, principal moments, changing origin, continuous systems

• Euler equations - stability analysis

• Convex functions and Legendre transformation
\[ \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} > 0 \]

• Hamiltonian, Hamilton’s equations
\[ p_i := \frac{\partial L}{\partial \dot{q}_i} \]
\[ H = \sum p_i \dot{q}_i - L \]
\[ \dot{q}_i = \dfrac{\partial H}{\partial p_i} \]
\[ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \]

- convexity implies equivalence between Lagrangian and Hamiltonian equations of motion

- Poisson brackets

\[ \{A, B\} = -\{B, A\} \]
\[ \{A, BC\} = \{A, B\}C + B\{A, C\} \]
\[ \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \]

- Time derivatives

\[ \dfrac{dF}{dt} = \{H, F\} + \dfrac{\partial F}{\partial t} \]

\[ D_FG := F, G \quad \dfrac{\partial F}{\partial t} = 0 \rightarrow F(t) = e^{-tD_H} F(t = 0) \]

- Canonical quantization; \( p_i, q_i \) become linear operator; functions of these operator satisfy

\[ [A, B] = i\hbar\{A, B\} \]

- \( \{z_i, z_j\} = J_{ij} \)

- Canonical transformations preserve \( J \)

\[ J_{ij} = \dfrac{\partial z_i'}{\partial z_k} J_{kl} \dfrac{\partial z_j'}{\partial z_l} \quad M_{ij} := \dfrac{\partial z_i'}{\partial z_k} \]

- Symplectic matrix \( MJM^t = J \)

- Differential forms - integration - exterior derivative \( (d, d^2 = 0) \) - closed forms \( (dF = 0) \) exact forms \( (F = dG) \)
• Canonical one form; canonical two form. \( \theta = \sum p_i dq_i \quad \omega = d\theta = \sum p_i \wedge dq_i \)

• Canonical transformations preserve the canonical 2 form

• Constructing canonical transformations using generating functions, \( S(q, P), S(q, Q) S(p, P) S(p, Q) \)

\[
A - B = dS \\ dA = dB = \omega
\]

• Time evolution canonical, integrals preserved under time evolution

\[
\int \omega, \quad \int \omega \wedge \omega \cdots \quad \int \omega \wedge \cdots \wedge \omega
\]

• Recurrence theorem

• \( z'_i = e^{D_F} z_i \) is a canonical transformation

• Hamilton Jacobi equation - equation for generating relating \( z_i(t) \) and \( z_i(0) \).

\[
\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0
\]

• Constrained Hamiltonian systems - when kinetic energy is not convex - give classical mechanics version of gauge transformations.

• Completely integrable systems

\[
\{F_i, F_j\} = \{H, F_i\} = 0 \quad dF_1 \wedge \cdots \wedge dF_N \neq 0
\]

• Torus theorem, action \( I_i \) - angle \( \Theta_i \) variables.
• Gravitational 2 body problem, conserved quantities, Kepler’s laws, scattering

• Gravitational 3 body problem, KAM theorem, small denominators, seek canonical transformation that preserved torus.

• Relativistic form of Newton’s second law.

• Mapping problems, logistical map