Lecture 15

Convenient states

\[ a = \frac{i}{\sqrt{2}} (q + ip) \quad p = \frac{i}{\sqrt{2}} (a^+ - a) \]
\[ a^t = \frac{i}{\sqrt{2}} (q - ip) \quad q = \frac{i}{\sqrt{2}} (a^+ + a) \]
\[ N = a^+ a \]
\[ 1z\rangle = e^{za^+} 10\rangle \]

Properties of 1z\rangle

1. \[ a 1z\rangle = a e^{za^+} 10\rangle = \]
\[ \sum_{n=0}^{\infty} \frac{Z^n}{n!} a(a^t)^n 10\rangle = \]
\[ \sum_{n=0}^{\infty} \frac{Z^n}{n!} (n(a^t)^{n-1} + (a^t)^n a) 10\rangle = \]
\[ \sum_{n=1}^{\infty} \frac{Z^n}{(n-1)!} (a^t)^{n-1} 10\rangle = \]
\[ Z \sum_{n=1}^{\infty} \frac{Z^{n-1}}{(n-1)!} (a^t)^{n-1} 10\rangle = \]
\[ Z \sum_{n=0}^{\infty} \frac{Z^n}{n!} a^n 10\rangle = \]
\[ Z1z\rangle \]

\[ 1z\rangle \] is an eigenstate of \( a \) with eigenvalue \( Z \) where \( Z \) is any complex number.
(2) Normalization

\[ \langle z' | z \rangle = \sum_{n=0}^{\infty} \frac{n!}{n! n!} (z^\ast z)^n = \sum_{n=0}^{\infty} \frac{(z^\ast z)^n}{n!} = e^{z^\ast z} \]

\[ \langle z' | z \rangle = e^{z^\ast z} \]

\[ \langle z | z \rangle = e^{-|z|^2} \]

\[ |z\rangle e^{-|z|^2/2} \text{ is normalized to unity.} \]

(3) Matrix element

\[ \frac{\langle z | a | z \rangle}{\langle z | z \rangle} = z \]

\[ \frac{\langle z | a^\dagger | z \rangle}{\langle z | z \rangle} = \frac{\langle z | a^\dagger | z \rangle}{\langle z | z \rangle} = z^\ast \]

\[ \frac{\langle z | p | z \rangle}{\langle z | z \rangle} = \frac{i}{\sqrt{2}} \frac{\langle z | (a^\dagger - a) | z \rangle}{\langle z | z \rangle} = \frac{2}{\sqrt{2}} \text{ Im}(z) = \sqrt{2} \text{ Im}(z) \]
\[
\frac{\langle Z_{19} | Z \rangle}{\langle Z_{12} \rangle} = \frac{i}{\sqrt{2}} \frac{\langle Z_{1} (a^{+}+a) | Z \rangle}{\langle Z_{1} | Z \rangle} = \frac{2}{\sqrt{2}} \text{Re}(\langle Z \rangle) = \sqrt{2} \text{Re} Z
\]

\[Z = \frac{1}{\sqrt{2}} \left( \langle q \rangle + i \langle p \rangle \right) \quad \langle p \rangle = \frac{1}{\sqrt{2}} (Z^{x}-Z)
\]
\[Z^{x} = \frac{1}{\sqrt{2}} \left( \langle q \rangle - i \langle p \rangle \right) \quad \langle q \rangle = \frac{1}{\sqrt{2}} (Z^{x}+Z)
\]

We can also calculate \((\Delta P)^{2} (\Delta q)^{2}\)

\[
\frac{\langle Z_{1} (|p\rangle - \langle p\rangle)^{2} | Z \rangle}{\langle Z_{1} | Z \rangle} = 
\]

\[
\frac{\langle Z_{1} (\frac{1}{\sqrt{2}} (a^{+}a))^{2} - \langle p^{2} \rangle | Z \rangle}{\langle Z_{1} | Z \rangle} = 1 + a^{+}a
\]

\[-\frac{1}{2} \left( \langle Z_{1} a^{+}a | Z \rangle + \langle Z_{1} a^{+}a | Z \rangle - \langle Z_{1} a^{+}a | Z \rangle - \langle Z_{1} a^{+}a | Z \rangle \right) \]
\[\langle Z_{1} | Z \rangle - \langle p \rangle \langle Z_{1} | Z \rangle \]

\[
\left( -\frac{1}{2} \left( (Z^{x})^{2} + (Z)^{2} - 2Z^{x}Z - 1 \right) - \langle p^{2} \rangle \right) \frac{\langle Z_{1} | Z \rangle}{\langle Z_{1} | Z \rangle}
\]

\[\text{Note:} \quad \langle p \rangle^{2} = \left( \frac{i}{\sqrt{2}} \right) (Z^{x} - Z)^{2} = -\frac{1}{2} (Z^{x} + Z^{x} - 2Z^{x}Z)
\]
\[= \langle p \rangle^{2} + \frac{1}{2} - \langle p \rangle^{2} = \frac{1}{2}
\]
\[(\delta p)^{2} = \frac{1}{2} \quad (\delta p) = \frac{1}{\sqrt{2}}\]
\[
\frac{\langle \mathbf{z} | (\mathbf{q} - \langle \mathbf{q} \rangle)^2 | \mathbf{z} \rangle}{\langle \mathbf{z} | \mathbf{z} \rangle} = \mathcal{S} \mathbf{q}^2 = \frac{1}{2}
\]

can be done using a analogous calculation.

Thus

\[
| \mathbf{q}, \mathbf{p} \rangle = \frac{| \mathbf{z} \rangle}{\langle \mathbf{z} | \mathbf{z} \rangle}^{1/2}
\]

is a unit normalized vector with

\[
\langle \mathbf{p} \rangle = \mathbf{p}, \quad \langle \mathbf{q} \rangle = \mathbf{q}, \quad \mathcal{S} \mathbf{p} = \mathcal{S} \mathbf{q} = \frac{1}{2}, \quad \mathcal{S} \mathbf{p} \mathcal{S} \mathbf{q} = \frac{1}{2}
\]

which means that it is a minimal uncertainty state

\[
(q-z) | \mathbf{q}, \mathbf{p} \rangle = 0 \quad \text{is the minimal uncertainty equation}
\]

\[
\left( \frac{1}{\sqrt{2}} (9-\langle q \rangle) + \frac{i}{\sqrt{2}} (p-\langle p \rangle) \right) | \mathbf{q}'\mathbf{p}' \rangle = 0, \quad \langle \mathbf{q} \rangle = q', \quad \langle \mathbf{p} \rangle = p'
\]

\[
(q-\langle q \rangle) | \mathbf{q}', \mathbf{p}' \rangle = i (p-\langle p \rangle) | \mathbf{q}', \mathbf{p}' \rangle \quad \mathcal{S} = \frac{\mathcal{S} \mathbf{p}}{\mathcal{S} \mathbf{q}} = 1
\]

\[
| \mathbf{q}, \mathbf{p} \rangle \quad \langle \mathbf{q'}, \mathbf{p'} | = \sqrt{\frac{1}{\pi}} e^{-\frac{1}{2}(q-q')^2 + i\mathbf{p} \cdot (q-q')}
\]

\[
\langle \mathbf{q}, \mathbf{p} | = \langle \mathbf{q} | \mathbf{z} \rangle \langle \mathbf{z} | \mathbf{p} \rangle
\]
Consider
\[ \int \langle q' | q'' \rangle \, dq'' \, dp'' \langle q'' | p'' | q' \rangle = \]
\[ \int \frac{1}{\sqrt{\pi}} e^{\frac{1}{2} (q-q'')^2 - \frac{1}{2} (q'-q'')^2 + i p''(q'\!-\!q')} \, dq'' \, dp'' \]

The $p''$ integral gives $2\pi \delta(q'\!-\!q') = 1$

\[ = \frac{2\pi}{\sqrt{\pi}} \delta(q'\!-\!q) \int e^{-\frac{1}{2} (q''-q')^2} \, dq'' \]
\[ = 2\pi \frac{1}{\sqrt{\pi}} \delta(q'\!-\!q) \int e^{-\frac{1}{2} q''^2} \, dq'' \]
\[ = 2\pi \frac{1}{\sqrt{\pi}} \delta(q'\!-\!q) \]
\[ = 2\pi \delta(q'\!-\!q) \]

or
\[ \delta(q'\!-\!q) = \int \langle q' | q'' \rangle \frac{dq'' \, dp''}{2\pi} \langle q'' | p'' | q' \rangle \]

\[ I = \int \frac{|q'' \rangle \, dq'' \, dp''}{2\pi} \langle q'' | p'' | 1 \rangle \]

This shows that any vector $|\psi\rangle$ can be written as a linear combination of minimal uncertainty states

\[ |\psi\rangle = \int \frac{|q'' \rangle}{2\pi} \langle q'' | p'' | 1 \rangle \]
This shows that the minimal uncertainty state are complete. In fact they are over complete.

We can write any one of them as a linear combination of the others with non vanishing expansion coeff.

Remark

(1) While it is possible to build a complete set of orthonormal states from $|10\rangle$ using transformed creation and annihilation operators, we have the alternative of working with states that have the narrow coordinate and momentum distributions.
Symmetries and conservation laws

There are certain correspondences that relate physical states where we expect that the results of all equivalent experiments will give equivalent results.

For example—consider our initial Stern-Gerlach experiment. We start with the magnets, and field gradients oriented in the \( \hat{z} \) direction; and measure \( \bar{s}_z \).

We could imagine a different experiment where the magnetic field and field gradients are oriented along the \( \hat{x} \) axis; and measure \( \bar{s}_x \). If the apparatus that created the initial beam is also rotated, then we expect that equivalent experiments have equivalent results.

\[
\begin{align*}
|z\rangle & \rightarrow |x\rangle \\
|\bar{z}\rangle & \rightarrow |\bar{x}\rangle \\
|c\rangle & \rightarrow |c\rangle' \\
S_z & \rightarrow S_x \\
\langle c|\bar{s}_z|c\rangle & = \langle c|\bar{s}_x|c\rangle'
\end{align*}
\]
The symmetry implies an underlying correspondence between states and operators that preserve all quantum mechanical observables.

\[ |a\rangle |b\rangle |c\rangle \rightarrow |a'\rangle |b'\rangle |c'\rangle \]

\[ O \ A \ B \rightarrow O' \ A' \ B' \]

\[ P_{a'b'} \frac{|\langle a|b\rangle|^2}{\langle a|a\rangle \langle b|b\rangle} \quad \frac{|\langle a'|b'|\rangle|^2}{\langle a'|a'|\rangle \langle b'|b'|\rangle} \]

\[ \langle a|b|a\rangle \quad \langle a|b|a'\rangle \quad \langle a'|b'|a'\rangle \]

\[ \langle B \rangle \quad \sum_{p_n} \frac{|\langle a_n|b|a_{n'}\rangle|^2}{\langle a_{n'}|a_{n'}\rangle} \quad \sum_{p_n} \frac{|\langle a'|b'|a_{n'}\rangle|^2}{\langle a_{n'}|a_{n'}\rangle} \]

Which kind of transformations on the states and operators preserve all of these fundamental observables.

Read Ch7 text

Wigner's Theorem

\[ |a\rangle \rightarrow |a'\rangle \quad \langle a|b\rangle = \langle a'|b'\rangle \}
\[ |b\rangle \rightarrow |b'\rangle \quad \langle a|b\rangle = \langle b'|a'\rangle \} \]
Wigner's theorem: The only correspondence that preserve all quantum probabilities are either

1. Linear and unitary
2. Antilinear and antiunitary

Case 1: Unitary

\[ |a\rangle = \sum \alpha_i |n_i\rangle <n_i|a\rangle \]

\[ U|n\rangle = |n'\rangle \]

\[ U|a\rangle = \sum \alpha_i |n_i\rangle <n_i|a\rangle \]

Case 2: Antiunitary

\[ T|n\rangle = |n'\rangle \]

\[ T\langle a| = T(\sum \alpha_i |n_i\rangle <n_i|a\rangle) = \]

\[ = \sum \alpha_i |n_i\rangle <n_i|a\rangle^* \]

\[ \langle T(|n\rangle |T(|a\rangle)\rangle = \langle \sum \alpha_i |n_i\rangle <n_i|a\rangle^* , \sum \alpha_i |n_i\rangle <n_i|a\rangle^* \rangle \]

\[ = \langle n_i|a\rangle <n_i|a\rangle^* \]

\[ = \langle n_i|a\rangle^* <n_i|a\rangle = <a|b\rangle = <b|c\rangle^* \]
any change of basis is unitary—
symmetries are active transformations
that correspond to specific
transformations

expectations for isolated systems

(1) invariance with respect to
translations

(2) invariance with respect to
rotations

(3) invariance with respect to
time (when experiment was performed

(4) invariance with respect to
velocity shifts

discrete symmetries

(1) invariance with respect to
space reflection

(2) invariance with respect to
time reversal

(3) gauge invariance.
It is instructive to consider the case of time translation symmetry.

\[ |a\rangle \quad U(t) \quad |a\rangle' \quad U(t) \quad |a\rangle'' \]
\[ |b\rangle \quad T(t) \quad |b\rangle' \quad T(t) \quad |b\rangle'' \]
\[ |c\rangle \quad T(t) \quad |c\rangle' \quad T(t) \quad |c\rangle'' \]
\[ |a\rangle \quad U(t+t') \quad |a\rangle'' \]
\[ |b\rangle \quad T(t+t') \quad |b\rangle'' \]
\[ |c\rangle \quad T(t+t') \quad |c\rangle'' \]

\[ |a\rangle'' = U(t')U(t)|a\rangle = U(t+t')|a\rangle e^{i\delta} \]
\[ |a\rangle'' = T(t')T(t)|a\rangle = T(t+t')(a\rangle) e^{i\delta} \]

For the case \( t = t' \) (antiunitary)

\[ |a\rangle'' = \sum |n\rangle^n<a^{\dagger n}|a\rangle = 2 \sum |n\rangle^n<a^{\dagger n} \]
\[ = 2 \sum |n\rangle^n<a^{\dagger n} \]
\[ = 2 \sum |n\rangle^n<a^{\dagger n} \]

These equations are inconsistent for continuous symmetries \( T(t) \) with \( T(0) = I \); \( T(t+\epsilon) = T(t)T(\epsilon) \)
The symmetry must be unitary. This means that the time shift operator

\[
U(t) = U(t)^+ \\
U(0) = I \\
U(t_1)U(t_2) = U(t_1 + t_2)
\]

It is possible that

\[
U(t_1)U(t_2) = U(t_1 + t_2)e^{i\theta(t_1,t_2)}
\]

It can be shown that in this case these phases can be absorbed by redefining \( U(t_1) = \) multiplying by a suitable phase. (see S Weinberg IV 12.104)

Note

\[
U(t_1+t_2)U^+(t_1+t_2) = I
\]

\[
\frac{dU}{dt}(t)U^+(t) = F(t)
\]

(i) \[
\frac{d}{dt}(uu^+) = \frac{d}{dt}(I) = 0 = \frac{dU}{dt}U^+ + U \frac{dU^+}{dt}
\]

\[
\frac{dU}{dt}U^+ = -U \frac{dU^+}{dt} = -\left(\frac{dU}{dt}U^+\right)^+,
\]

(ii) \[
F(t+c) = \frac{dU}{dt}(t+c)U^+(t+c) = \frac{dU}{dt}(t)u(t)cU^+(t)U(t)
\]

\[
= \frac{dU}{dt}(t)U^+(t) = F(t).
\]
Thus we see
\[ F(t) = F(0) = -F^+(0). \]

Let \( F = -iH \), \( H = H^+ \)

\[
\frac{dU(t)}{dt} U^+(t) = F(t) = F(0) = -iH
\]

\[
\frac{du}{dt} = -iH U(t) \quad U(0) = I
\]

This gives an equation for the unitary time evolution operator \( U(t) \)

It has the formal solution
\[
U(t) = e^{-iHt} = \sum_{\text{basis}} e^{-iH_n t} |n\rangle \langle n|.
\]

How are equivalent states at different times related?

\[ |c(t)\rangle = U(t) |c(0)\rangle \]

\[ \langle n | c(t) \rangle = \langle n | U(t) | c(0) \rangle \]

\[
\frac{d}{dt} \langle n | c(t) \rangle = \langle n | \frac{d}{dt} U(t) | c(0) \rangle = -i \langle n | H U(t) | c(0) \rangle
\]

= \langle n | H U(t) | c(0) \rangle
\[
\frac{d}{dt} \langle n | c(t) \rangle = -i \langle n | H | c(t) \rangle = -i \frac{\hbar}{m} \langle n | H | m \rangle \langle m | c(t) \rangle
\]

for states labeled by continuous variables.

\[
\frac{d}{dt} \langle q | c(t) \rangle = -i \int \langle q | H | q' \rangle dq' \langle q' | c(t) \rangle
\]

The quantity \( H \) is called the generator of time evolution—\( \) or the Hamiltonian.

The requirement \( H \) is independent of \( t \) is valid for isolated systems when \( t \) is time \( H \) has units of \( 1/\text{time} \). Normally we associate \( H \) with energy \( \Rightarrow H \leftrightarrow \frac{E}{\hbar} \).

\( \hbar \) is Planck's constant—\( \) which has units of energy \( x \) time.
This equation is the Schrödinger equation.