Lecture 17

Last time time translation symmetry

\[ |a(t)\rangle = U(t)|a(0)\rangle \]

\[ U(0) = I \]

\[ U(t_2)U(t_1) = U(t_1+t_2) \]

From these conditions we found

\[ U(t) = e^{-iHt/\hbar} \]

where \( H = H^\dagger \) is independent of \( t \).

\( H \) is called the Hamiltonian of the theory.

This is the solution of

\[ \hbar \frac{dU}{dt} = HU \quad U(0) = I \]

\( \hbar \) is Planck's constant / \( 2\pi \) - it has units Energy x time = momentum x distance

In the Schrödinger picture

\[ |\Psi(t)\rangle = U(t)|\Psi(0)\rangle \]

\[ \Psi(x,t) = \langle x|U(t)|\Psi(0)\rangle \]
\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \langle x | i\hbar \frac{\partial}{i\hbar} u(t) | \psi(t) \rangle \]

\[ = \int \langle x | H(t) | \psi(\tau) \rangle d\tau \langle \psi(\tau) | u(t) | \psi(t) \rangle \]

\[ = \int \langle x | H(t) | \psi(t) \rangle d\psi(t) \psi(t) \]

This equation is the Schrödinger equation; \( \langle x | u(t) | \psi(t) \rangle \) is the Schrödinger wave function. \( \langle x | \psi(t) \rangle \) defines the basis used to represent this equation.

We can use any other basis; for example, in the stationary uncertainty basis \( \langle x | \rightarrow \langle \ell | \rangle \)

\[ \psi(t) \rightarrow \langle n | \psi(t) \rangle = \langle n | u(t) | \psi(t) \rangle = \psi_n(t) \]

\[ i\hbar \frac{\partial}{\partial t} \psi_n(t) = \langle n | i\hbar \frac{\partial}{i\hbar} u(t) | \psi(t) \rangle \]

\[ = \frac{\hbar}{2} \langle n | H(t) | \ell \rangle \langle \ell | u(t) | \psi(t) \rangle \]

\[ = \frac{\hbar}{2} \langle n | H(t) | \ell \rangle \psi_{\ell}(t) \]
we can use this time dependence to determine the time evolution of the expectation values of observables

$$\langle \psi(t) | A | \psi(t) \rangle =$$

$$\langle \psi(0) | U(t)^\dagger A U(t) | \psi(0) \rangle$$

at this point it is useful to consider an example

Consider a spin $\frac{1}{2}$ system

basis $| \uparrow \rangle = | \uparrow \rangle_2 \quad 1- \rangle = | \downarrow \rangle_2$

Hamiltonian

$$H = - \vec{\mu} \cdot \vec{B} = - \frac{e}{mc} \frac{\hbar}{2} \sigma_z \cdot B$$

$$= - \frac{e\hbar B}{mc} \sigma_z$$

initial state $| \psi \rangle \rightarrow \text{ components}$

$$\langle + | \psi \rangle = c_+ \langle 0 |$$

$$\langle - | \psi \rangle = c_- \langle 0 |$$

initial state
Time dependent state

\[ |C(t)\rangle = U(t) |C(0)\rangle \]

\[ C_+(t) = \langle +1 | C(t) \rangle = \langle +1 | U(t) | C(0) \rangle \]

\[ C_-(t) = \langle -1 | C(t) \rangle = \langle -1 | U(t) | C(0) \rangle \]

\[ i \hbar \frac{dC_+}{dt} = \langle +1 | H | U(t) | C \rangle = \langle +1 | H | + \rangle C_+(t) + \langle +1 | H | - \rangle C_-(t) \]

\[ i \hbar \frac{dC_-}{dt} = \langle -1 | H | U(t) | C \rangle = \langle -1 | H | + \rangle C_+(t) + \langle -1 | H | - \rangle C_-(t) \]

\[ \langle +1 | H | + \rangle = -\langle -1 | H | - \rangle = \frac{e\hbar B}{2mc} \quad (-1) \]

\[ \langle +1 | H | - \rangle = \langle -1 | H | + \rangle = 0 \]

This gives the equations

\[ i \hbar \frac{dC_+}{dt} = -\frac{e\hbar B}{2mc} C_+(t) \]

\[ i \hbar \frac{dC_-}{dt} = \frac{e\hbar B}{2mc} C_-(t) \]

\[ \frac{dC_+}{dt} = i \frac{eB}{2mc} C_+(t) \quad \omega = \frac{eB}{2mc} \]

\[ \frac{dC_-}{dt} = -\frac{eB}{2mc} C_-(t) \]

\[ C_+(t) = e^{i\omega t} C_+(0) \]

\[ C_-(t) = e^{-i\omega t} C_-(0) \]
Let's use these solutions to find the time evolution of $S_x$ in this state:

$$\langle S_x(t) \rangle = \langle C(t) / S_x / C(t) \rangle =$$

$$= \left( c_+^* c_-^* \right) \frac{k}{2} \left( \begin{array}{c} 0^2 \\ \iota \end{array} \right) \left( \begin{array}{c} c_+^*(t) \\ c_-^*(t) \end{array} \right)$$

$$= \frac{k}{2} \left( c_+^*(t) c_-(t) + c_-^*(t) c_+(t) \right)$$

$$= \frac{k}{2} \left( e^{-2i\omega t} \left( c_+^*(t) c_-(t) + e^{2i\omega t} c_-^*(t) c_+(t) \right) \right)$$

$$= \frac{k}{2} \left\{ \cos(2\omega t) \cdot Z \Re \left( c_+^*(t) c_-(t) \right) \right. \right.$$

$$+ \sin 2\omega t \cdot Z \Im \left( c_+^*(t) c_-(t) \right) \left. \right\}$$

which shows that in general, this expectation values oscillates with frequency $2\omega = \frac{eB}{mc}$.

The Schrödinger picture is not the only approach to compute the time dependence of expectation values.
Consider

$$\langle \psi(t) | \mathcal{A} | \psi(t) \rangle =$$

$$\langle \psi(t) | U(0)^\dagger A U(t) | \psi(t) \rangle$$

$$\langle \psi(t) | \mathcal{A}(t) | \psi(t) \rangle$$

where

$$\mathcal{A}(t) = U(0)^\dagger A U(t)$$

$$= e^{i \frac{Ht}{\hbar}} A e^{-i \frac{Ht}{\hbar}}$$

This is called the Heisenberg picture. In this approach the time dependence appears in the operator rather than the vector.

Obviously, the expectation values remain the same.

$$\frac{d \mathcal{A}}{dt} = e^{i \frac{Ht}{\hbar}} (i \frac{H}{\hbar}) A e^{-i \frac{Ht}{\hbar}}$$

$$- e^{i \frac{Ht}{\hbar}} A (i \frac{H}{\hbar}) e^{-i \frac{Ht}{\hbar}}$$

$$= \frac{i}{\hbar} [H, \mathcal{A}(t)]$$
These equations of motion are called the Heisenberg equations of motion.

The abstract equations are most useful in a basis

\[ \langle n | A^{\dagger} m \rangle = A_{nm} \]

\[ \frac{dA_{nm}}{dt} = \frac{i}{\hbar} \frac{g}{k} \left( H_{km} A_{nm} - A_{nm} H_{km} \right) \]

\[ A_{nm}(t_0) = \text{matrix elements of the time independent operators.} \]

Let us consider our example of a spin \( \frac{1}{2} \) particle in a magnetic field using the Heisenberg equations of motion

\[ \frac{dS^{x+}}{dt} = \frac{i}{\hbar} \left( -\frac{e\mathbf{B}}{2mc} \right) \left( \sigma^{x+} S^{x+} + \sigma^{-} S^{-} - \sigma^{+} S^{+} - \sigma^{-} S^{-} \right) \]

\[ \left( \sigma = \sigma_z \right) \]

\[ \frac{dS^{x-}}{dt} = \frac{i}{\hbar} \left( -\frac{e\mathbf{B}}{2mc} \right) \left( \sigma^{x-} S^{x+} + \sigma^{+} S^{+} - \sigma^{-} S^{-} - \sigma^{x-} \sigma^{-} S^{x-} \right) \]
\[
\frac{dS_{x+}}{dt} = \frac{i}{\hbar} \left( -\frac{eB}{2mc} \right) \left( S_{++} + S_{+-} - S_{-+} - S_{--} \right)
\]

\[
\frac{dS_{x-}}{dt} = \frac{i}{\hbar} \left( -\frac{eB}{2mc} \right) \left( S_{+-} + S_{--} - S_{-+} - S_{++} \right)
\]

\(\sigma_{++} = -\sigma_{--} = 1\);  \(\sigma_{+-} = \sigma_{-+} = 0\);  for the
Heisenberg equations we cannot make assumptions about \(S_{xii}\):

\[
\frac{dS_{x++}}{dt} = -i \frac{eB}{2mc} \left( S_{x++} - S_{x+} \right) = 0
\]

\[
\frac{dS_{x+-}}{dt} = -i \frac{eB}{2mc} \left( S_{x+-} + S_{x-} \right) = -i \frac{eB}{mc} \sigma_{x+-}
\]

\[
\frac{dS_{x-+}}{dt} = -i \frac{eB}{2mc} \left( -S_{x-+} - S_{x+} \right) = i \frac{eB}{mc} \sigma_{x-+}
\]

\[
\frac{dS_{x--}}{dt} = -i \frac{eB}{2mc} \left( -S_{x--} + S_{x-} \right) = 0
\]

From these equations we see \(S_{x++}\)
and \(S_{x--}\) are independent of \(t\);
Since they vanish initially they vanish for all time.
\[ S_{x+} = e^{-i \frac{eBt}{mc}} S_{x+}(\psi) = e^{-i \frac{eBt}{mc}} \frac{\hbar}{2} \]
\[ S_{x-} = e^{i \frac{eBt}{mc}} S_{x-}(\psi) = e^{i \frac{eBt}{mc}} \frac{\hbar}{2} \]

To compute the expectation value of \( S_x \)

\[ \frac{\hbar}{2} \begin{pmatrix} C^*_+(\psi) & C^*_-(\psi) \end{pmatrix} \begin{pmatrix} 0 & e^{-i \frac{eBt}{mc}} \\ e^{i \frac{eBt}{mc}} & 0 \end{pmatrix} \begin{pmatrix} C_+(\psi) \\ C_-(\psi) \end{pmatrix} = \]
\[ \frac{\hbar}{2} \begin{pmatrix} C^*_+(\psi) C_+(\psi) e^{-i \frac{eBt}{mc}} + C^*_-(\psi) C_-(\psi) e^{i \frac{eBt}{mc}} \end{pmatrix} \]

\[ \hbar \text{Re}(C^*_+(\psi) C_+(\psi)) \cos \left( \frac{eBt}{mc} \right) + \]
\[ \hbar \text{Im}(C^*_+(\psi) C_+(\psi)) \sin \left( \frac{eBt}{mc} \right) \]

This is exactly the same result that we obtained by solving the Schrödinger equation.

The interesting thing about the Heisenberg equations is their close relation to classical mechanics.
The equations are

\[ \frac{dA}{dt} = \frac{i}{\hbar} \left[ H, A \right] \]

\[ \frac{d\bar{P}}{dt} = \frac{i}{\hbar} \left[ H, \bar{P} \right] \quad \text{time evolution of pairs of complementary operators} \]

\[ \frac{d\bar{\varphi}}{dt} = \frac{i}{\hbar} \left[ H, \bar{\varphi} \right] . \]

Recall that any operator can be expressed in terms of complementary operators.

Classical Hamiltonian mechanics

\[ H \rightarrow H(\bar{\varphi}, \bar{P}) \]

Equations of motion

\[ \frac{dP_i}{dt} = -\frac{\partial H}{\partial q_i} = -\frac{\partial}{\partial q_j} \left( \frac{\partial H}{\partial q_i \partial p_j} - \frac{\partial H}{\partial p_j \partial q_j} \right) \]

\[ = -\{H, P_i\} \]

\[ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} = -\{H, q_i\} \]

\( P_i(t) = \text{canonical transform} \)
\[ q_i(t) = e^{-i[H, \frac{1}{\hbar}]} q_i(0) \quad \text{canonrical trans.} \]

where \[ \{H, \cdot \} \] acts like a differential operator

\[ \{q_i, p_j\} = \delta_{ij} \quad \frac{dp_i}{dt} = -\{H, p_i\} = -\frac{\hbar}{i} \{H, \frac{1}{\hbar}\} e^{-i[H, \frac{1}{\hbar}]} \]
\[ \{q_i, q_j\} = 0 \quad \frac{dq_i}{dt} = -\{H, q_i\} \]
\[ \{p_i, p_j\} = 0 \quad \frac{dp_i}{dt} = \frac{i}{\hbar} \{H, p_i\} e^{-i[H, \frac{1}{\hbar}]} \]

\[ \{q_i, p_j\} = i\hbar \delta_{ij} \quad \frac{dp_i}{dt} = \frac{i}{\hbar} \{H, p_i\} = \frac{1}{\hbar} \{H, \frac{1}{\hbar}\} \]
\[ \{p_i, q_j\} = 0 \quad \frac{dq_i}{dt} = \frac{i}{\hbar} \{H, q_i\} \]
\[ \{p_i, p_j\} = 0 \]

\[ \{\delta x, \delta y\} \to -\frac{i}{\hbar} \{\delta x, \delta y\} \]

This replacement provides a strong analogy between classical and quantum mechanics.

The biggest difference is that in quantum mechanics \( qp \neq pq \) while in classical mechanics \( pq = qp \).

This correspondence with classical mechanics provides clues in how to construct Hamiltonians.
Here is one more picture in addition to the Schrödinger and Heisenberg pictures that is useful in applications. It is called the interaction picture.

Recall that in our treatment of the Heisenberg picture we wrote down the matrix equations using a time-independent basis.

In the interaction picture we assume

\[ H = H_0 + H_I \]

\[ |C_o(t)\rangle = e^{-iH_0t/H} |C_o(0)\rangle. \]

\[ \langle C(0)\mid U^+_I(t) A U(t) \mid C(0)\rangle = \]

\[ \langle C(0)\mid U^+_o(t) U_{01}(t) U^+_I(t) A U(t) U_o(t) U_{10}(t) \mid C(0)\rangle \]

\[ \langle C_o(t)\mid U_{10}(t) U^+_I(t) A U(t) U^+_o(t) \mid C_o(t)\rangle, \]

\[ A_I(t) = U_{01}(t) U^+_I(t) A U(t) U^+_o(t) \]

\[ \frac{dA_I}{dt} = U_o(t) \left( -\frac{i}{\hbar} (H_0 - H) \right) U^+_I(t) A U(t) U^+_o(t) - U_o(t) U^+_I(t) A U(t) \left( -\frac{i}{\hbar} (H_0 - H) \right) U^+_o(t) \]
\[ H_I(t) = U_0(t) H_I U_0^\dagger(t) \]

\[ \frac{dA_I}{dt} = \frac{i}{\hbar} \left( [H_I(t), A_I(t)] - A_I(t) H_I(t) \right) \]

\[ \frac{dA_I}{dt} = \frac{i}{\hbar} \left[ H_I(t), A_I(t) \right]. \quad A_I(0) = A \]

The new feature in the interaction picture is that
\[ H_I(t) = U_0(t) H_I U_0^\dagger(t) \]

is now a time dependent operator:

\[ A_I(t) = A + \frac{i}{\hbar} \int_0^t \left[ H_I(t'), A_I(t') \right] dt' \]

\[ H_I(t') = e^{i H_I t'} + i \frac{H_I}{\hbar} e^{i H_I t'} \]

If we include a basis of eigenstates of \( H_0 \), the matrix elements of \( H_I(t') \) in these states is equivalent to using

\[ \langle c(0) e^{i H_I t'} | H_I e^{-i H_I t'} | c(0) \rangle = \langle c(t) | H_I | c_0(t) \rangle. \]