Lecture 2

Last time we introduced "filters" associated with devices that manipulate spin states in the Gell-Mann experiment.

\[ |+\rangle \langle +| \]

\[ |-\rangle \langle -| \]

\[ 0 \]

\[ I = |+\rangle \langle +| + |-\rangle \langle -| \]

\[ \langle +|+\rangle = \langle -|-\rangle = 1 \]

\[ \langle +|-\rangle = \langle -|+\rangle = 0 \]

\[ 0: |x\rangle \langle x|1 = 0 \]

\[ 1: |x\rangle \langle x|1 = |x\rangle \langle x|1 \]

Addition of objects is distributive and commutative.

We used a magnetic field \( B \) to gradient to rotate spin.

\[ |+\rangle \langle -| ; |-\rangle \langle +| \]

We found that the order of the products matter.
we introduced
\[ [x, y] = xy - yx \]
\[ \{x, y\} = xy + yx \]
\[ xy = \frac{1}{2} [xy] + \frac{1}{2} \{xy\} \]

also note objects not a division ring
\[ AB = 0 \quad A \neq 0 \Rightarrow B = 0 \]
\[ 1 + x < 1 \quad 1 + y = 0 \quad A \neq 0 \quad B \neq 0 \]

so far we have only introduced integers

physically rotate the field

gradient so it makes an angle
\[ \theta \]
with with the \( z \) axis

then we can use it to make a new device denoted by
\[ 1 + i \theta \times \theta \]
\[ 1 - i \theta \times \theta \]

we have by analogy
\[ 1 + \theta \times \theta \]
\[ 1 - \theta \times \theta \]
\[ 1 + i \theta \times \theta \]
\[ 1 - i \theta \times \theta \]
\[ 1 + i \theta \times \theta + 1 - i \theta \times \theta = 1 \]
consider the experiment

\[ \begin{array}{c}
\text{2} \quad \text{0} \quad \text{2} \\
\end{array} \]

\[ |+\rangle |+\rangle = |+\rangle |+\rangle |+\rangle = |+\rangle (1 \langle 0 | 0 \rangle + |0 \rangle \langle 0 | 1 \rangle) |+\rangle \langle +| \]

assuming \( \langle \alpha | \beta \rangle \) is a number

subtracting

\[ (1 - \langle +| 0 \rangle \langle 0 | +\rangle - \langle +| 0 \rangle \langle 0 | +\rangle) |1\rangle \langle +| = 0 \]

\[ |1\rangle = \langle +| 0 \rangle \langle +| 0 \rangle + \langle +| 0 \rangle \langle 0 | 1 \rangle \]

how do we interpret these quantities?

1. If \( \Theta = \frac{\pi}{2} \), \( \langle +| 0 \rangle = +1 \) and

\[ \langle +| 0 \rangle \langle +| 0 \rangle = 1 \]

\[ \langle +| 0 \rangle \langle 0 | 1 \rangle = 0 \]

2. If \( \Theta = -\frac{\pi}{2} \)

\[ \langle +| 0 \rangle \langle -| 1 \rangle = 1 \]

\[ \langle +| 0 \rangle \langle +| 0 \rangle = 0 \]

in these cases everything on nothing passes through the triple filter.
for \( \hat{O} = \Pi_{2} \) (\( \chi \alpha \hat{a} \) )

\[ |+\rangle <+1+\hat{O}> <+1+\hat{O}> |+\rangle \]

sometimes we will measure +
sometimes we will measure nothing

similarly

\[ |+\rangle <+1-\hat{O}> <-1+\hat{O}> |+\rangle \]

will have similar properties. We don't know what will happen - but we expect the numbers observed will be the same.

\[ N = N_{+} + N_{-} \]
\[ N_{+} = N <+1+\hat{O}> <+1+\hat{O}> \]
\[ N_{-} = N <+1-\hat{O}> <-1-\hat{O}> \]

If this interpretation is correct then we expect

\[ <+1\hat{O}> <+1\hat{O}> = P_{+} \] probability of seeing \( Z^{+} \) after measuring \( X^{+} \)

\[ <+1\hat{O}> <-1\hat{O}> = P_{-} \] probability of seeing \( Z^{+} \) after measuring \( X^{-} \) starting from \( Z^{-} \)
for this to be true $\langle +10^+ \rangle \langle 0^+1^+ \rangle$
must be real and positive.

If this is true for all $\langle 1b \rangle = 0$
either

(1) $\langle +10^+ \rangle = \langle 0^+1^+ \rangle$ is real
(2) $\langle +10^+ \rangle = \langle 0^+1^+ \rangle^*$ is complex.

More generally, A B operators
with symbols $\langle n_0 \rangle \langle m_0 \rangle$ $\langle m_b \rangle \langle n_b \rangle$

$$P_m = \langle n_0 \rangle \langle m_b \rangle \langle m_b \rangle \langle n_0 \rangle \geq 0$$

$$\sum P_m = 1$$

Returning to the spin $\frac{1}{2}$ case

$$N = N_+ + N_-$$

The symbol

$$B = \frac{\hbar}{2} \langle 0^+ \rangle \langle 0^+1 \rangle - \frac{\hbar}{2} \langle 0^- \rangle \langle 0^-1 \rangle$$

$$\langle 1^+ \rangle \langle 1^+ \rangle \langle B \rangle \langle 1^+ \rangle \langle 1^+ \rangle = \frac{\hbar}{2} (P_+ - P_-) \langle 1^+ \rangle \langle 1^+ \rangle$$

is $\langle 1^+ \rangle \langle 1^+ \rangle \times$ the average value of
the magnetic moment. To be consistent with $\theta = 2 \times -2 = 1$

$$P_+ + P_- = 1$$
$$P_+ - P_- = \cos \theta$$
Experimentally one finds this is true:

\[ P_+ = \frac{1}{2} (1 + \cos \omega) = \cos^2 \frac{\omega}{2} \]
\[ P_- = \frac{1}{2} (1 - \cos \omega) = \sin^2 \frac{\omega}{2} \]

Thus for \( \langle a|c \rangle = \langle b|b \rangle = 1 \)

\[ P_{ab} = \langle a|b \rangle \langle b|a \rangle \]

System starts in state \( a \) of \( A \); \( B \) is measured giving \( b \) - the probability that the system will still be found in state \( a \) of \( A \) is completely symmetrical.

\[ P_{ba} = \langle b|a \rangle \langle a|b \rangle = \langle a|b \rangle \langle b|a \rangle = P_{ab} \]

Is \( \langle a|b \rangle \) real?
<a> \rightarrow \text{ puts system in a state where measurement of } A \text{ gives value } 'a' \text{.} \\
\langle a \mid \rightarrow \text{ picks out part of state where measurement of } A \text{ gives value } 'a' \text{.} \\
\text{It is useful to work with these objects:} \\
\langle a \rangle = \sum l_{bn} \langle b_{n} | a \rangle = \text{<ket> vector} + \\
\langle a \rangle = \sum a_{bn} \langle b_{bn} | a \rangle = \text{dual vector} \\
\langle c | = \sum c_{bn} \langle b_{bn} | c \rangle \\
\langle c | a \rangle = \sum c_{bn} \langle b_{bn} | a \rangle = \sum c_{bn} l_{bn} \langle b_{bn} | a \rangle \\
\text{suggests:} \\
l_{bn} \text{ is a basis for } | l \rangle \text{ vector (ketis)} \\
\langle b_{bn} | \text{ is a basis for } < l | \text{ vector (bras)} \\
\text{quantities } < b_{bn} | a \rangle \text{ are called wave functions of } | a \rangle \text{ in the basis } l_{bn} >
apply to

\[ |x^+\rangle = |+\rangle <+| x^+\rangle + |-\rangle <-1 x^+\rangle \]
\[ |x^-\rangle = |+\rangle <+1 x^-\rangle + |-\rangle <-1 x^-\rangle \]

de these vectors must be orthogonal and normalized and \( \langle x^+ | x^- \rangle = \frac{1}{\sqrt{2}} \).

This can be satisfied if

\[ |x^+\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \]
\[ |x^-\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \]

we also have

\[ |y^+\rangle = |+\rangle <+1 y^+\rangle + |-\rangle <-1 y^+\rangle \]
\[ |y^-\rangle = |+\rangle <+1 y^-\rangle + |-\rangle <-1 y^-\rangle . \]

These have the same properties — in addition they must satisfy

\[ \langle x^\pm | y^\pm \rangle = \frac{1}{\sqrt{2}} \]

These conditions cannot be satisfied if \( \langle a | b \rangle \) are real.
\[ |\psi_1\rangle = (\alpha |1+\rangle + \beta |1-\rangle) \]

\[ \langle a|c \rangle = \alpha \langle a|1+\rangle + \beta \langle a|1-\rangle \]

\[ \langle c|a \rangle = \langle c|a \rangle^* = \alpha^* \langle 1|c \rangle + \beta^* \langle 1|c \rangle \]

\[ = (\alpha^* \langle 1| + \beta^* \langle -1|) |c \rangle \]

\[ \langle c|1 = \alpha^* \langle 1| + \beta^* \langle -1| \]

\[ \langle c|a \langle a|c \rangle = \left( \frac{1}{\sqrt{2}} \langle 1| + \frac{i}{\sqrt{2}} \langle -1| \right) \left( \frac{1}{\sqrt{2}} |1+\rangle + \frac{i}{\sqrt{2}} |1-\rangle \right) = \frac{1}{2} + \frac{i}{2} = 1 \]

\[ \langle c|a \langle a|c \rangle = \left( \frac{1}{\sqrt{2}} \langle 1| - \frac{i}{\sqrt{2}} \langle -1| \right) \left( \frac{1}{\sqrt{2}} |1+\rangle - \frac{i}{\sqrt{2}} |1-\rangle \right) = \frac{1}{2} - \frac{i}{2} = 0 \]

\[ \langle c|a \langle a|c \rangle = \left( \frac{1}{\sqrt{2}} \langle 1| + \frac{i}{\sqrt{2}} \langle -1| \right) \left( \frac{1}{\sqrt{2}} |1+\rangle + \frac{1}{\sqrt{2}} |1-\rangle \right) = \frac{1}{2} + \frac{i}{2} = \frac{\sqrt{2} + i}{2} \]

\[ |1\rangle \pm \frac{i}{2} = \frac{1}{2} \sqrt{|1+|} = \frac{1}{\sqrt{2}} \]

These combinations have all of the desired properties.
Thus we conclude that $\langle a|b \rangle$
must be complex.

Let $A = \langle a|a \rangle$, $C = \langle b|b \rangle$, $B = K\langle a|b \rangle$,

\[
\langle a - r a b|a - r a b \rangle = \\
\langle a|a \rangle + r^2 \langle b|b \rangle - 2 r \langle a|b \rangle - r^2 \langle b|a \rangle \geq 0
\]

$A + r^2 \cdot C - 2 r B \geq 0 \quad \forall r \, \text{real}$

$C = 0 \Rightarrow B = 0 \Rightarrow K\langle a|b \rangle \leq K\langle a|a \rangle \langle b|b \rangle$.

$C \geq 0 \Rightarrow r = B/C$

$A + B^2/C - 2B^2/C \geq 0$

$AC = B^2 \Rightarrow K\langle a|c \rangle \langle b|b \rangle \geq K\langle a|b \rangle$

This means that $\langle a|b \rangle$ satisfies the Schwarz inequality and our space of kets is a complex inner product space (Hilbert space).
Operators

\[ |a\rangle \langle b| \text{ simplest example.} \]

Take

\[ (|a\rangle \langle b|) |c\rangle = |a\rangle \langle b| |c\rangle \]

\[ (|a\rangle \langle b|) (\lambda |c\rangle + |d\rangle) = \lambda |a\rangle \langle b| |c\rangle + |a\rangle \langle b| |d\rangle \]

Linear operators on the space of particles

\[ S_z = \frac{\hbar}{2} |+\rangle \langle +| - \frac{\hbar}{2} |\rangle \langle -| \]

\[ S_\theta = \frac{\hbar}{2} |0\rangle \langle 0| + \frac{\hbar}{2} |\rangle \langle -| \]

\[ \langle a| S_\theta |c\rangle = \frac{\hbar}{2} \langle a| 10 \rangle \langle 10| - \frac{\hbar}{2} \langle a| 1\rangle \langle 1| \]

for \[ \langle a|a\rangle = 1 \]

This gives the value of a number of measurements of \[ S_z \] for initial and final states in the state \[ |c\rangle \]

If \[ a \] is not normalized

\[ \frac{\langle a| S_\theta |c\rangle}{\langle a|a\rangle} = \text{same quantity} \]
properties of \( S_6 \)

\[
S_6 \ 10^+ > = \cdot \ S_6 \ 10^+ > =
\]

\[
= \left( \frac{k}{2} \ 10^+ \times 0^+ \ 1 - \frac{k}{2} \ 10^- \times 0^- \ 1 \right) \ 10^+ >
\]

\[
= \frac{k}{2} \ 10^+ >
\]

\[
(S_6 - \frac{k}{2}) \ 10^+ > = 0.
\]

We also have

\[
(S_6 + \frac{k}{2}) \ 10^- > = 0.
\]

Finally

\[
(S_6 + \frac{k}{2}) (S_6 - \frac{k}{2}) \ 1c > =
\]

\[
(S_6 + \frac{k}{2}) (S_6 - \frac{k}{2}) (10^+ \times 0^+ 1c > + 10^- \times 0^- 1c > ) =
\]

\[
0 + (-k) \ (S_6 + \frac{k}{2}) \ 10^+ 1c > \ 10^- 1c > = 0.
\]

* \( 10^+ > \) is an eigenvector of \( S_6 \)
with eigenvalue \( \frac{k}{2} \)

\( 10^- > \) is an eigenvector of \( S_6 \)
with eigenvalue \( \frac{-k}{2} \)

The set of all eigenvectors of \( S_6 \)
are a basis.

\( 10^+ > 10^- 1 + 10^- > 10^+ = I \)

\( < 0^+ 10^- > = 0 \)
since $\pm \frac{\hbar}{2}$ correspond to values of measurements of our magnetic moment $\vec{\mu}$, the eigenvalues of $\hat{S}_z$ must be real.

**General case**

\[
\begin{align*}
A &= \sum a_n |n\rangle \langle n| \\
I &= \sum |n\rangle \langle n| \\
a_n &= a_n^* \\
\langle n | m \rangle &= \delta_{mn}
\end{align*}
\]

We have

1. \[(A - a_n)|n\rangle = 0\]
   - eigenvectors make a basis
   - eigenvalues real

2. \[
\prod_{n \neq m} (A - a_n) |n\rangle = 0
\]
   \[
\prod_{n \neq m} (a_m - a_n) |m\rangle
\]
   \[
\prod_{n \neq m} \left( \frac{A - a_n}{a_m - a_n} \right) |c\rangle
\]

Linear operators with these properties are called self-adjoint.
\[ \prod_{n \neq m} \left( \frac{A - a_n}{a_m - a_n} \right) \leq |k \rangle \langle k | \ c \rangle = \]

\[ \prod_{n \neq m} \frac{(a_m - a_n)}{(a_m - a_n)} \quad |m \times m | c \rangle = \]

\[ |m \rangle \langle m | c \rangle. \]

Since \( |c \rangle \) is arbitrary,

\[ |m \rangle \langle m | = \prod_{n \neq m} \left( \frac{A - a_n}{a_m - a_n} \right) = \text{Polynomial in } A. \]

\[ I = \sum |m \times m | \]

\[ A = \sum |m \rangle a_m \langle m | \]

\[ A^n = \sum |m \rangle (a_m)^n \langle m | \]

\[ f(A) = \sum |m \rangle f(a_m) \langle m | \quad \text{(any func) } \]

\[ = \sum_{m} f(a_m) \prod_{n \neq m} \frac{(A - a_n)}{(a_m - a_n)} = P(A) \]

any function of \( A \) can be expressed as a polynomial in \( A \).
\[<b|A|b> = \sum <b|n> a_n <n|b> = \sum K|b|n> |^2 a_n = \sum P_n a_n.\]

\[<b|A|b> = <b|A|b>^*\]

**Adjoint operator**

\[|b> = A|a>\]

we define \(A^+\) by

\[<c|b> = <c|A|a>\]

\[<b|c> = <c|A|^* = <a|A^+|c>\]

for self adjoint \(A\)

\[<c|A|a> = \sum <c|n> a_n <n|a> = \sum K|a> |^2 a_n <n|c> = \sum <n|c> a_n^* <c|n> = \sum <a|n> a_n^* <n|c>\]

\[\text{but } a_n = a_n^*\]

\[a|2|ln>a_n <n|c> = A^+ = A\]
comments on observables.

Normally one works backwards = observables = self adjoint operators = prove properties

measurements = properties = self adjoint

There may not be experiments that can measure every self adjoint operator. There are more limited by how they are related to classical quantities.

\[
\frac{\langle a_1 b \rangle \langle b_1 c \rangle}{\langle a_1 c \rangle \langle b_1 b \rangle} = p_{ab} \quad \text{prob}
\]

\[
\langle a_1 A_1 a_1 \rangle = \frac{2}{n} |\langle a_1 | a_1 n \rangle|^2 \quad \text{an expectation value}
\]

If we do a statistically significant number of measurements \( \langle a_1 \rightarrow p_n \langle a_1 n \rangle \)

\( p_n \) are classical props -

\[
\langle A \rangle \_p = \sum p_n \langle a_1 A_1 a_1 \rangle = \\
\text{Tr} ( \langle A_1 a_1 n \rangle p_n \langle a_1 n \rangle) \\
\text{Tr} (A \_p)
\]
If \( \rho = |a\rangle \langle a| \) \( \text{Tr}(\rho A) = \langle a| A |a\rangle \)

\( \text{Tr}(\rho) = \langle a|a\rangle = 1 \)

\( \text{Tr}(\rho^n) = 1 \) \{ pure state \}
proof

\[ A = \langle a | c \rangle \quad B = K | a b \rangle \quad c = r^* \quad ||a|| = 1 \]

\[ C = \langle b | b \rangle \]

\[ \langle a - r a b | a - r a b \rangle \geq 0 \]

\[ \langle a | c \rangle + r^2 \langle b | b \rangle - r \alpha \langle b | a c \rangle - r \alpha \langle a | c b \rangle \]

choose \( \alpha = \frac{K(a | b)}{\langle a | b \rangle} \)

\[ \alpha^* \langle b | c \rangle = \alpha \langle a | b \rangle = \langle a | b \rangle = 1 \langle a | b \rangle \]

\[ \langle a | c \rangle + r^2 \langle b | b \rangle - 2r \langle a | b \rangle \geq 0 \]

if \( \langle b | b \rangle = 0 \) then becomes negative for large \( r \) unless \( \langle a | b \rangle = 0 \)

\[ \langle a | c \rangle \cdot 0 \geq 0 \]

if \( \langle b | b \rangle \neq 0 \) choose \( r = \frac{K(a | b)}{\langle b | b \rangle} \)

\[ \langle a | c \rangle + \frac{K(a | b)}{\langle b | b \rangle} \cdot 2r \frac{K(a | b)}{\langle b | b \rangle} \geq 0 \]

\[ \langle b | b \rangle \cdot r \cdot \langle a | c \rangle \geq \langle a | b \rangle \cdot r \cdot \langle b | b \rangle \]

The quantity \( \langle a | b \rangle \) is called a complex inner product. A complex vector space with an inner product is called a Hilbert space (for a dimensions we require an additional property called completeness which we discuss later).