More on symmetries

\[
\begin{pmatrix}
R & \tilde{u} & a \\
0 & 1 & t \\
0 & 0 & 1
\end{pmatrix} = g_l \begin{pmatrix}
R_l & \tilde{u}_l & a_l \\
0 & 1 & t_l \\
0 & 0 & 1
\end{pmatrix}
\]

\[U(q_2)U(q_1) = e^{i\frac{\theta(q_2,q_1)}{4}}U(q_2q_1)\]

Rotations, translations, and time translations can redefine phases so \(\theta = 0\).

Recall time translations

\[U(t_2)U(t_1) = U(t_2+t_1)\]
\[U(0) = I\]

\[i\frac{dU(t)}{dt}U^+(t)\]

is (1) self-adjoint
(2) independent of \(t\)

Space translations in 1 direction

\[U(q_1)U(q_1) = U(q_2+q_1)\]
\[U(0) = I\]

Rotations about fixed axis

\[U(\theta_2)U(\theta_1) = U(\theta_1+\theta_1)\]
\[U(0) = I\]
\[U(\phi) = U(\phi+2\pi)\]
Following what was done in the case of time translations:

\[ U(a) = e^{i p \cdot a / \hbar} \quad p = p^+ \text{ indep of } a \]

\[ U(\phi) = e^{i j \cdot \phi / \hbar} \quad j = j^+ \text{ indep of } \phi \]

since we want to consider arbitrary axes we write:

\[ p \rightarrow \tilde{p} \cdot \hat{n} \] (translations in \( \hat{n} \) direction)

\[ j \rightarrow \tilde{j} \cdot \hat{n} \] (rotations about \( \hat{n} \) axis)

\[ \langle x|\psi \rangle = \text{state centered about } \bar{x} = 0 \]

\[ \langle x|U(a)|\psi \rangle = \langle \bar{x}+a|\psi \rangle \]

\[ = \text{state centered about } \bar{x} = -a \]

\[ \langle 4|U(a)|\bar{x} \rangle = \langle 4|\bar{x}+a \rangle \]

In this to hold for all \( |\psi \rangle \)

\[ U(a)|x\rangle = |x-a\rangle \]

Denote the eigenvalue of the operator \( x \) by \( x' \)

\[ x U(a)|x'\rangle = x|x-a\rangle = (x'-a)|x'-a\rangle \]

\[ = (x'-a)U(a)|x\rangle \]
This means that \( u(x) \ket{x'} \) is an eigenstate of \( x \) with eigenvalue \( x' + a \).

We also have

\[
    u^+(x) x u(x) \ket{x'} = (x' - a) u^+(x) u(x) \ket{x'}
    = (x' - a) \ket{x'} = (x - a) \ket{x'}
\]

since this holds for all \( x' = 0 \),

\[
    u^+(x) x u(x) = x - a
\]


\[
    u(x) x u^+(x) = x + a
\]

We note

\[
    e^{-i \frac{\hbar}{\lambda} p a} x e^{i \frac{\hbar}{\lambda} p a} = x + a
\]

differentiate both sides with respect to \( a \) set \( a = 0 \)

\[
    \frac{i}{\hbar} (p x - x p) = +1
\]

\[
    [p, x] = -i \hbar
\]
\[ P \langle x | p' \rangle = p' \langle x | p' \rangle \]

\[ P = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad (\text{representation of } E_p, x \gamma = -i \hbar \quad \text{in } x \times \text{representation}) \]

\[ \frac{\hbar}{i} \frac{d \langle x | p' \rangle}{\langle x | p' \rangle} = p' \, dx \]

\[ \langle x | p' \rangle = e^{ -\frac{i}{\hbar} x \cdot p' } \]

\[ \delta (x - x') = \int \langle x | p' \rangle d p' \langle p'(x') = \]

\[ = \int e^{ \frac{i}{\hbar} (x - x') \cdot p } (c l^2 \, d p \quad u = p l_n \]

\[ = \int e^{ i u (x - x') } (c l^2 \, k \, du \]

\[ = (c l^2 \, \hbar \, \delta (x - x') \Delta n \quad c = \frac{1}{\sqrt{2 \pi n}} \]

\[ \langle x | p' \rangle = \frac{1}{\sqrt{2 \pi n}} e^{ i \frac{p' \cdot x}{\hbar} } \]

\[ \langle p | x' \rangle = \frac{1}{\sqrt{2 \pi n}} e^{ -i \frac{p \cdot x'}{\hbar} } \]

because \( K |x | p' \rangle = K | p | x' \rangle = \sqrt{2 \pi n} \) for all \( x, p \) are complementary observables.

\[ U(\alpha) \bar{\psi} U^\dagger(\alpha) = \bar{x} + \bar{\alpha} \]

\[ \begin{pmatrix} +i q \cdot x / \hbar & -i q \cdot x / \hbar \end{pmatrix} \begin{pmatrix} \bar{p} \bar{c} \end{pmatrix} = \bar{p} - \bar{\alpha} \]

\[ V(\xi) P V^\dagger(\xi) = \bar{p} - \bar{\xi} \]
\[ V(q) \text{ is the momentum translation operator.} \]

\[ \frac{dX}{dt} = \frac{i}{\hbar} [H, X] \]

\[ [H, X] = 0 \quad X \text{ conserved} \]

\[ U(t) X U^*(t) = X \]

Both translations and rotations satisfy

\[ U(t) U(\tilde{\phi}) U^*(t) = U(\tilde{\phi}) \]

\[ U(t) U(\tilde{\phi} \hat{\theta}) U^*(t) = U(\tilde{\phi} \hat{\theta}) \]

\[ U(\tilde{\phi} \hat{\theta}) U(\tilde{\psi} \hat{\theta}) U^*(\tilde{\phi} \hat{\theta}) = U(\tilde{\phi} \tilde{\theta}) \]

These equations lead to

\[ [H, \tilde{J}] = [H, \tilde{P}] = 0 \quad [P_i, P_j] = 0 \]

These first 2 equations mean that \( \tilde{P}, \tilde{J} \) are conserved quantities.

For this reason

\[ H = \text{energy} \]

\[ \tilde{P} = \text{linear momentum} \]

\[ \tilde{J} = \text{angular momentum} \]
which are classically conserved quantities. The factor $\hbar$ gives both $H$ and $\hat{P}$ the correct units.

Not all symmetries are continuous. Standard examples are space reflection and time reversal:

\[
P^2 = I \quad \Rightarrow \quad P = P^{-1}
\]
\[
T^2 = I \quad \Rightarrow \quad T = T^{-1}
\]
\[
P \times P^{-1} = -\mathbf{x}
\]
\[
\hat{P} \cdot \hat{P}^{-1} = -\hat{\mathbf{p}}
\]
\[
\hat{P} \cdot \hat{J} \cdot P^{-1} = \hat{J} \quad (\hat{J} = \mathbf{x} \times \hat{\mathbf{p}})
\]
\[
T \times T^{-1} = \mathbf{x}
\]
\[
T \cdot \hat{P} \cdot T^{-1} = -\hat{P}
\]
\[
T \cdot \hat{J} \cdot T^{-1} = -\hat{J}
\]

\[
P \cdot \nabla \cdot P^{-1} = \nabla \quad \text{pseudo vector} \quad (\hat{J})
\]
\[
P \cdot A \cdot P^{-1} = -A \quad \text{pseudo scalar} \quad (\hat{J}, \frac{\mathbf{p}}{m} + \frac{\hat{p}}{m} \cdot \hat{J})
\]

If $H$ contains pseudoscalar operators then $PHP^{-1} \neq H$.

It turns out that the weak Hamiltonian does not commute with the space reflection operator.
Time reversal

\[ T U(t) T^{-1} = U(-t) \]

\[ T e^{i \mathbf{H} t} T^{-1} = e^{-i \mathbf{H} t} = e^{i \mathbf{H} t} = 0 \]

\[ T \mathbf{(iH)} T^{-1} = -i \mathbf{H} \]

There are 2 possibilities:

\[ T \] is unitary \( \Rightarrow \) \( T \mathbf{H} T^{-1} = -\mathbf{H} \)
\[ T \] is anti-unitary \( \Rightarrow \) \( T \mathbf{H} T^{-1} = \mathbf{H} \)

\[ \mathbf{H} \xi' \rangle = \mathbf{E} \xi' \rangle \]

\[ \mathbf{H} (\mathbf{T} \xi) \rangle = \mathbf{T} \mathbf{T}^{-1} \mathbf{H} \mathbf{T} \xi \rangle = \mathbf{E} \mathbf{T} \xi \rangle \]

In the unitary case every eigenvector with energy \( \mathbf{E} \) is paired with another with eigenvalue \(-\mathbf{E}\). If \( \mathbf{H} \) is not bounded from above then it can't be bounded from below! This suggests that the universe could be unstable with respect to small perturbations.

\( T \) is taken to be anti-unitary so can construct operators that do not commute with \( T \).

\( (\mathbf{p} \mathbf{J} + \mathbf{J} \mathbf{p}) \)
the weak interaction contains terms that violate time reversal invariance.

Discrete translations

\[
\begin{align*}
T(a) \times T(a) &= x + a \\
T(a) V(x) T^*(a) &= V(x + a) = V(x)
\end{align*}
\]

Periodic potentials are invariant with respect to translations along lattice planes through a multiple of the lattice spacing:

\[
H = \frac{p^2}{2m} + V(x)
\]

\[
T(a) H T^*(a) = H
\]

\[
H |E\rangle = E' |E\rangle = \delta
\]

\[
H T(a) |E\rangle = T(a) H |E\rangle = E' T^*(a) |E\rangle.
\]

\[
(T(a))^n |E\rangle
\]
is an eigenvalue of \(H\) with eigenvector \(E'\).

define \( |n\rangle = (T(a))^n |E\rangle\), \(n = -\infty \ldots \infty \)
\( T(a) \left| n \right> = \left| n+1 \right> \) by def.

The states \( \left| n \right> \) are not eigenstates of \( T(a) \). But

\[
\left| \omega \right> = \sum_{n=-\infty}^{\infty} e^{i n \omega} \left| n \right>
\]

\[
H \left| \omega \right> = \sum_{n=-\infty}^{\infty} e^{i n \omega} H \left| n \right> = E \left| \omega \right>
\]

\[
T(a) \left| \omega \right> = \sum_{n=-\infty}^{\infty} e^{i n \omega} \left| n+1 \right> =
\]

\[
e^{-i \omega} \sum_{n=-\infty}^{\infty} e^{i(n+1)\omega} \]

\[
= e^{-i \omega} \sum_{n=-\infty}^{\infty} e^{i n \omega} \left| n \right>
\]

\[
= e^{-i \omega} \left| \omega \right>
\]

\( \left| \omega \right> \) is a simultaneous eigenstate of \( T(a) \) and \( H \).

\[
\langle x \mid T(a) \left| \omega \right> = e^{-i \omega} \langle x \mid \omega \rangle = \langle x-a \mid \omega \rangle
\]

\[
e^{-i \omega} \langle x \mid \omega \rangle = \langle x-a \mid \omega \rangle.
\]

Try a solution of the form

\[
\langle x \mid \omega \rangle = e^{ikx} u_k(x), \quad k = \frac{\omega}{a}
\]
\(-i \theta + ikx\) \\
\(e^{ik(x-\alpha)} U_{\alpha}(x) = e^{ik(x-\alpha)} \quad \theta = +k\alpha\) \\
\(U_{\alpha}(x) = U_{\alpha}(x-\alpha)\)

This shows that \\
\(\langle x|k\alpha \rangle = e^{ikx} U_{\alpha}(x) =\)

plane wave \times periodic function.

This is called Bloch's theorem.

Potentials\(s\)

\[ i\hbar \frac{d\psi}{dt} = (T + V) \psi \]

\(\langle \psi_n(t) \rangle = e^{-i\hbar E_n t} \langle \psi_n(0) \rangle \)

let \( V = V + V_0 \quad V_0 = \text{constant} = 1 \)

\(\langle \psi_n(t) \rangle = e^{-i\hbar (E_n + V_0) t} \langle \psi_n(0) \rangle \)

state vectors differ by a time dependent phase

Next consider the case that \( V \) is \emph{in} in a finite time: \\
\(\langle \psi_n(t) \rangle = e^{-i\int_0^t V_0 dt} - \frac{i}{\hbar} E_n \langle \psi_n(0) \rangle \)

\[ \frac{d}{dt} \langle \psi_n(t) \rangle = -\frac{i}{\hbar} (V_0(t) + E_n) \langle \psi_n(t) \rangle \]
This clearly satisfies the equation. The wave functions differ by the phase
\[ e^{i\frac{\hbar}{U} \int_0^t V(x) \, dt} = e \]

Beam of charged particles - split inside of 2 conductors at different potentials - no fields, no force recombine - there is a phase difference that depends on the elapsed time in the conductors.

Phase depends on \( \hbar \) - quantum mechanical effect.