Lecture 27

Discrete symmetries

Space reflection \( P \)

Time reversal \( T \)

Expectations \( P^2 = I \quad T^2 = I \)

\[
\begin{align*}
P \times P' &= -X \\
P \cdot P' &= -\beta \\
P \times P' &= -E
\end{align*}
\]

\[
\begin{align*}
T \times T' &= X \\
T \cdot T' &= -\beta \\
T \times T' &= -E
\end{align*}
\]

\( P \bar{V} P' = \bar{V} \) \( \bar{V} \) is a pseudo vector

\( P \bar{S} P' = -\bar{S} \) \( \bar{S} \) is a pseudo scalar

The scalar product of a pseudo vector and vector is a pseudo scalar.

The Hamiltonian for the weak interaction contains some pseudo-scalar contributions. This means that energy eigenstates do not come in pairs related by space reflection.

Time reversal \( T \)

\[
T U(t) T^{-1} = U(-t) = e^{it\hat{H}t} = e^{it(\text{im})T^{-1}t} \]

\( i\tilde{H} = T(i\tilde{H})T^{-1} \)
We have 2 options by Wigner's thm.

1. $THT'^{-1} = -H \quad T \text{ unitary}$
2. $THT'^{-1} = H \quad T \text{ antiunitary}$

The first option is ruled out for Hamiltonians that have a spectrum unbounded from above because

$$H|E\rangle = E|E\rangle$$

$$THT'^{-1}|E\rangle = E T|E\rangle = -H T|E\rangle$$

$$H(T|E\rangle) = -E(T|E\rangle)$$

which means that the spectrum of the Hamiltonian is unbounded from below.

This means that the theory is unstable with respect to small perturbations.
periodic potentials (here \( U(\alpha) \) is translation by \( \alpha \))

\[
U(\alpha) V(x) U^\dagger(\alpha) = V(x+\alpha)
\]

\[
H = \frac{p^2}{2m} + V(x)
\]

\[
U(\alpha) H U^\dagger(\alpha) = H
\]

If \( |\psi\rangle \) is an eigenfunction of \( H \) with eigenvalue \( E \)

\[
H \, |\psi\rangle = E \, |\psi\rangle = 0
\]

\[
H \, U(\alpha) \, |\psi\rangle = U(\alpha) \, H \, U(\alpha) \, |\psi\rangle =
\]

\[
= U(\alpha) \, H \, |\psi\rangle
\]

\[
= U(\alpha) \, E \, |\psi\rangle
\]

\[
= E \, U(\alpha) \, |\psi\rangle
\]

\[
H \, (U(\alpha) \, |\psi\rangle) = E \, U(\alpha) \, |\psi\rangle
\]

Multiplying by \( U(\alpha) \) gives another eigenstate with the same energy.

Likewise for any \(-\infty < n < \infty\)

\[
H \, (U(\alpha)^n \, |\psi\rangle) = E \, (U(\alpha)^n \, |\psi\rangle)
\]

define \( |n\rangle \equiv (U(\alpha)^n \, |\psi\rangle \)
we see that this system is infinitely degenerate:

$$U(c) |n\rangle = |n+1\rangle$$

$|n\rangle$ is not an eigenstate of $U(c)$; however $E U(c) |n\rangle = 0$ so we should be able to find simultaneous eigenstates of both.

Try

$$|\phi\rangle = \sum_{n=-\infty}^{\infty} e^{i\alpha n} |n\rangle$$

$$U(c) |\phi\rangle = \sum_{n=-\infty}^{\infty} e^{i\alpha n} U(c) |n\rangle$$

$$= \sum_{n=-\infty}^{\infty} e^{i\alpha n} |n+1\rangle$$

$$= e^{-i\phi} \sum_{m=-\infty}^{\infty} e^{i\alpha m} |m\rangle$$

$$= e^{-i\phi} |\phi\rangle$$

These are eigenstates of $H$ and $U(c)$ in any value of $\alpha$. 

it is interesting to consider the basis wave function \( \psi(0) \)

\[
\langle x | \psi(0) \rangle = e^{-i \theta} \langle x | \psi \rangle
\]

assume that

\[
\langle x | \psi \rangle = e^{-i k x} u(x) \
\]

\[
e^{-i k(x+a)} u(x+a) = e^{-i \omega - i k x} u(x)
\]

\[
e^{-i(ka-\omega)} u(x+a) = u(x)
\]

if we choose \( \omega = k a \), \( k = 0 a \),

then we see

\[
\langle x | \psi \rangle = e^{-i \omega x/a} u(x)
\]

where \( u(x) = u(x+a) \)

This result is called Bloch's theorem - simultaneous eigenstates of \( H \) and \( H \) (periodic) are plane waves multiplied by periodic functions.
Potentials - quantum effects

In classical mechanics adding a constant to a potential does not change any forces and has no measurable effects.

Here we consider what happens in quantum mechanics:

\[ \frac{i \hbar}{\partial t} |\psi(t)\rangle = (H + V_0) |\psi(t)\rangle \]

If \( H_0 |n\rangle = E_n |n\rangle \) then \( H |n\rangle = (E_n + V_0) |n\rangle \).

The change in potential shifts all energies:

\[ |\psi(t)\rangle = \left( \begin{array}{c} c_n e^{-iE_n t/\hbar} |n\rangle \\ 2 \left( c_n e^{-iE_n t/\hbar} |n\rangle \right) e^{-iV_0 t/\hbar} \end{array} \right) \]

We see that the inclusion of the constant potential in the Hamiltonian leads to an additional time dependent phase.
Next consider a potential that is constant in \( x \) but a possible function of time \( t \):

\[ H = H_0 + V(t) \]

\[ i\hbar \frac{d|\psi\rangle}{dt} = (H_0 + V(t))|\psi\rangle \quad (1) \]

if we know the solution to

\[ i\hbar \frac{d|\psi\rangle}{dt} = H_0 |\psi\rangle \]

then the solution to (1) is clearly

\[ |\Psi(t)\rangle = e^{-\frac{i}{\hbar} \int_{t_0}^{t} V(t') dt'} \]

This reduces to the previous result when the potential is constant.

Differentiating \( |\psi\rangle \) we have

\[ i\hbar \frac{d}{dt} \left( |\psi\rangle e^{-\frac{i}{\hbar} \int_{t_0}^{t} V(t') dt'} \right) = \]

\[ (H_0 - i\hbar \frac{1}{\hbar} V(t)) (|\psi\rangle e^{-\frac{i}{\hbar} \int_{t_0}^{t} V(t') dt'}) \]
consider a beam of charged particles

\[ \text{split the beam into 2 beams} \]

\[
\begin{array}{c}
\text{conducting cylinders} \\
\text{let each beam pass through a different conducting cylinder} \\
\text{turn on the potential difference} V_0 \text{ after the particles enter the conductor and off before they leave.} \\
\text{the wave functions differ by} \\
\langle \psi_0(t) \rangle = \langle \psi(t) \rangle e^{-i \int_0^t V(t') dt'} = \langle \psi_0(t) \rangle e^{i \phi(t)} \\
\phi = -\frac{i}{\hbar} \int_0^t V(t') dt' \\
\text{when these are combined} \\
\langle \psi(t) \rangle = \langle \psi_0(t) \rangle \frac{1}{2} \left( 1 - e^{i \phi} \right) 
\end{array}
\]
\[ \Delta \phi = \frac{mg \theta_2 T \sin s}{h} \]

\( s \) is angle made with a perfectly horizontal plane

\[ T = l_1/l_2 = \frac{m_1 m_2}{m_2} = l_2^2/l_1 \]

\[ \Delta \phi = \frac{m^2 g l_2 \sin s}{h^2} \]

One can observe oscillations as \( s \) is varied:

\[ \sim \sin^2 \left( \frac{1}{h^2} m^2 g l_2 \sin s \right) \]

The important observation is that this effect depends on \( m^2/h^2 \) — even though \( m \) does not enter the classical equations.

R. Colella / A. Overhauser / S. A. Werner — observe using neutron interferometry.
\[ \langle \psi(t) | \psi(t) \rangle = \frac{1}{i} \left( 1 - e^{i\phi} \right) \left( 1 - e^{-i\phi} \right) \]
\[ = \frac{1}{i} \left( 2 - 2\cos \phi \right) \]
\[ = \frac{1}{2} \left( 1 - \cos \phi \right) \]
\[ = \sin^2(\phi/2) \]
\[ = \sin^2 \left( \frac{\hbar}{\hbar} \int_0^{t'} V(t') dt' \right) \]

We see that we an interference that depends on \( V(t) \) and \( T \). This is a quantum mechanical effect because of the \( \hbar \) dependence.

Example: gravity induced quantum interference.

\[ m \ddot{\omega} = -mgh \quad \text{(mass drops out classically)} \]

\[ \left( -\frac{\ddot{x}}{2m} + mg \right) \psi = i\hbar \frac{d}{dt} \psi \]

\[ \left( -\frac{\hbar^2}{2m} \nabla^2 + mq \right) \psi = i\hbar \frac{d}{dt} \psi \]

\[ \left( -\frac{\hbar^2}{2m} \nabla^2 + q \right) \langle x | \psi \rangle = i\hbar \frac{d}{dt} \langle x | \psi \rangle \]

We see that the mass explicitly appears in the Schrödinger equation in the combination \( m/\hbar \).
Aharonov-Bohm effect

\[ \mathbf{B} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\nabla \phi - \frac{\hbar}{c} \frac{\partial \mathbf{A}}{\partial t} \]

\[ \phi \rightarrow \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \]

\[ \mathbf{A} \rightarrow \mathbf{A} + \nabla \lambda \]

Physical fields do not depend on the explicit choice of \( \lambda \), \( \phi \), and \( \mathbf{A} \).

Schrödinger equation for a particle in a field is

\[
\left( \frac{\mathbf{p} - \frac{e}{c} \mathbf{A}}{2m} + e \phi \right) |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle
\]

Under gauge transformation: time invariance

\[
\left( \left( \frac{\mathbf{p} - \frac{e}{c} \mathbf{A}}{2m} - \frac{e}{c} \nabla \phi \right)^2 + e \phi \right) |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle
\]

Observe

\[
e^{-ie\lambda/\hbar c} \left( \mathbf{p} - \frac{eA}{c} - \frac{e\nabla \phi}{c} \right)^2 e^{ie\lambda/\hbar c} = \left( \mathbf{p} - \frac{eA}{c} \right)^2
\]

This shows that the solution of these two equations:

\[
e^{-ie\lambda/\hbar c} \left( \frac{\mathbf{p} - \frac{eA}{c}}{2m} + e \phi \right) e^{ie\lambda/\hbar c} = i\hbar \frac{\partial |\psi\rangle}{\partial t}
\]
\[ |\Psi \rangle = |\Psi \rangle e^{i e A / mc} \]

classically, for a particle in a magnetic field

\[ L = \frac{m}{2} \dot{x}^2 + \frac{e}{c} \dot{x} \cdot \mathbf{A} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = m \ddot{x}_i + \frac{e}{c} \left( \partial_j A_j \dot{x}_i - \dot{x}_j \partial_j A_i \right) \]

note

\[ F = \mathbf{\nabla} \times \mathbf{B} = \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{A}) = \mathbf{\nabla} (\mathbf{\nabla} A_i) - \mathbf{\nabla} \cdot \mathbf{\nabla} A \]

\[ m \ddot{x}_i = (\mathbf{\nabla} \times \mathbf{B})_i \]

In a Feynman path integral

the magnetic field gives a phase

\[ e^{-\frac{i}{\hbar} \int \frac{e}{c} \mathbf{A} \cdot d\mathbf{x}} \]

consider

\[ (S_{x_1} + S_{x_2}) \frac{i e}{\hbar c} \mathbf{A} \cdot d\mathbf{x} = \frac{i e}{\hbar c} \Phi \]

a flux enclosed by the 2 curves.
In these 2 paths there is a phase difference \( e^{i \frac{\Phi}{\hbar}} \) where \( \Phi \) is the flux in the cylinder.

The path integral is a sum of terms with 1 phase plus one with the other phase.