Lecture 3

Last time

$$\langle \pm 1 \pm \rangle_0 \langle \pm 1 \pm \rangle_2 = \begin{cases} \cos^2 \frac{\theta}{2} \\ \sin^2 \frac{\theta}{2} \end{cases}$$

when $\theta = \frac{\pi}{2}$ these both give $\frac{1}{2}$

In general

$$0 \leq \langle a|b \rangle \langle b|a \rangle \leq 1$$

represent probabilities which means that they are numbers between 0 and 1.

For these to be positive for any $|a\rangle, |b\rangle$ ($\langle a|a \rangle = \langle b|b \rangle = 1$)

we must have

$$\langle a|b \rangle = \langle b|a \rangle \quad \langle a|b \rangle \text{ real}$$

$$\langle a|b \rangle = \langle b|a \rangle^* \quad \langle a|b \rangle \text{ complex}$$

From these observations we determine that $\langle a|b \rangle$ must be complex.
to show this consider

\[ |+\rangle_x = |1\rangle \langle +| + \langle +|-\rangle_x \]

\[ \langle +1 \rangle_x = \langle +1| + \langle +1|-2 \rangle_x \]

\[ 1 = \langle +1+ \rangle_x = \left( \begin{array}{c}
\langle +1+ \rangle_x \\
\langle +1- \rangle_x
\end{array} \right) \]

\[ = \frac{\langle +1+ \rangle_x + \langle +1- \rangle_x}{2} \]

we begin by assuming that \( \langle c|b \rangle \) are always real. It follows that

\[ \langle +1+ \rangle_z = \frac{1}{\sqrt{2}} \]

\[ \langle +1- \rangle_z = \frac{1}{\sqrt{2}} \]

if we do the same calculation with \( l \rightarrow x \), \( l \rightarrow x \) we also find

\[ \langle -1+ \rangle_z = \frac{1}{\sqrt{2}} \]

\[ \langle -1- \rangle_z = \frac{1}{\sqrt{2}} \]
the requirement $\langle +1\{-1\} \rangle_x = 0$ means that one of the factors has a different sign than the other three.

Example

$$1^+ \rangle_x = \frac{1}{\sqrt{2}} \; 1^+ \rangle_z + \frac{1}{\sqrt{2}} \; 1^- \rangle_z$$

$$\langle +1 \rangle_x = \frac{1}{\sqrt{2}} \; \langle +1 \rangle_z + \frac{1}{\sqrt{2}} \; \langle -1 \rangle_z$$

$$1^- \rangle_x = \frac{1}{\sqrt{2}} \; 1^+ \rangle_z - \frac{1}{\sqrt{2}} \; 1^- \rangle_z$$

$$\langle -1 \rangle_x = \frac{1}{\sqrt{2}} \; \langle +1 \rangle_z - \frac{1}{\sqrt{2}} \; \langle -1 \rangle_z$$

Calculating

$$\langle +1^+ \rangle_x = \frac{1}{2} + \frac{1}{2} = 1$$

$$\langle -1^- \rangle_x = \frac{1}{2} + \frac{1}{2} = 1$$

$$\langle -1^+ \rangle_x = \frac{1}{2} - \frac{1}{2} = 0$$

as expected. The problem is that $y$ is also $\perp$ to $z$ and $x$, the same analysis gives

$$1^+ \rangle_y = \frac{1}{\sqrt{2}} \; 1^+ \rangle_z + \frac{1}{\sqrt{2}} \; 1^- \rangle_z$$

$$1^- \rangle_y = \frac{1}{\sqrt{2}} \; 1^+ \rangle_z - \frac{1}{\sqrt{2}} \; 1^- \rangle_z$$
Comparing the expressions for $\langle \pm \uparrow \rangle_\gamma$ we see

\[
\begin{align*}
\langle \uparrow \uparrow \rangle_\gamma &= 1 & \langle \uparrow \downarrow \rangle_\gamma &= 0 \\
\langle \downarrow \uparrow \rangle_\gamma &= 0 & \langle \downarrow \downarrow \rangle_\gamma &= 1
\end{align*}
\]

Since the angle between the $x$ and $y$ axis is $\pi/2$ both of these should be $\pm 1/\sqrt{2}$.

There are 3 more possibilities; but the result is the same in all cases. This rules out the assumption that $\langle a|b \rangle$ can be taken as real.

Let us consider the complex option $\langle a|b \rangle = \langle b|a \rangle^*$. Try

\[
\begin{align*}
1+ \gamma_x &= 1+ \gamma_x \langle \uparrow \uparrow \rangle_\gamma + \frac{1}{\sqrt{2}} \langle \uparrow \downarrow \rangle_\gamma \\
1- \gamma_x &= 1+ \gamma_x \langle \downarrow \uparrow \rangle_\gamma + \frac{1}{\sqrt{2}} \langle \downarrow \downarrow \rangle_\gamma \\
1+ \gamma_y &= 1+ \gamma_y \langle \uparrow \uparrow \rangle_\gamma + \frac{1}{\sqrt{2}} \langle \uparrow \downarrow \rangle_\gamma \\
1- \gamma_y &= 1+ \gamma_y \langle \downarrow \uparrow \rangle_\gamma + \frac{1}{\sqrt{2}} \langle \downarrow \downarrow \rangle_\gamma
\end{align*}
\]
(Remark since $|\alpha\rangle$ corresponds to a state $e^{iq}\delta\langle\alpha|$ corresponds to the same state. There is some freedom in how we choose overall phases.)

with the expressions on the last page

\[
\begin{align*}
\langle +1 | +1 \rangle_x & = \left| \left( \frac{1}{\sqrt{2}} \langle +1 | \right)_x \right|^2 + \left| \left( \frac{1}{\sqrt{2}} \langle -1 | \right)_x \right|^2 = 1 \\
\langle -1 | -1 \rangle_x & = \left| \left( \frac{1}{\sqrt{2}} \langle +1 | \right)_x \right|^2 + \left| \left( \frac{1}{\sqrt{2}} \langle -1 | \right)_x \right|^2 = 1 \\
\langle +1 | -1 \rangle_x & = \left( \frac{1}{\sqrt{2}} \langle +1 | \right)_x \left( \frac{1}{\sqrt{2}} \langle -1 | \right)_x = \frac{1}{2} \end{align*}
\]

consider for example

\[
\begin{align*}
\langle +1 | & = \langle +1 | +1 \rangle_x + \langle +1 | -1 \rangle_x \\
& = \left( \frac{1}{\sqrt{2}} \langle +1 | \right)_x + \left( \frac{1}{\sqrt{2}} \langle -1 | \right)_x \\
\langle +1 \pm \rangle_x & = \left( \frac{1}{\sqrt{2}} \langle +1 | \right)_x \left( \frac{1}{\sqrt{2}} \langle +1 \pm \rangle_x \right) + \left( \frac{1}{\sqrt{2}} \langle -1 | \right)_x \left( \frac{1}{\sqrt{2}} \langle -1 \pm \rangle_x \right) = \frac{1}{2} \mp \frac{i}{2} \\
\left| \langle +1 \pm \rangle_x \right|^2 & = \left( \frac{1}{2} \mp \frac{i}{2} \right) \left( \frac{1}{2} \pm \frac{i}{2} \right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\end{align*}
\]

as required similar calculations hold for $\langle -1 \pm \rangle_x$
thus we conclude that in order to treat spin \( \frac{1}{2} \) particles in 3 dimensions we need to use complex numbers.

Summary

- \( \overline{S} \overline{\theta} \) measured quantity
- \( \pm \frac{\hbar}{2} \) outcomes of measurements
- \( |\pm\rangle \) "state vector" (initial)
- \( \langle\pm\rangle \) "state vector" (final)
- \( \langle a|b\rangle \) complex number.

\[ S_\theta = \langle 1+\rangle_0 \frac{\hbar}{2}_0 \langle +| + \rangle + \langle 1-\rangle_0 \langle -| \rangle_0 \langle -1 \rangle \]

\( |a\rangle \), \( |b\rangle \) represent the same physical state.

\[ P_{ab} = |K|_ab|^2 = \frac{\langle a| b\rangle\langle b| a\rangle}{\langle a| a\rangle\langle b| b\rangle} \]

probability that a state prepared in \( |a\rangle \) after measuring \( |b\rangle \) remains in \( |a\rangle \)

\[ \langle a|S_\theta|a\rangle = \text{mean value of } S_\theta \text{ in state } |a\rangle. \]
\[ I = 1 + \gamma_0 \leq 1 + 1 - \gamma_0 < 1 \] every measurement of \( \gamma_0 \) can have 1 of 2 possible values.

**General Considerations**

A, B, C measurable quantities with \( N \) possible values

\[ a_1, \ldots, a_N \quad \text{possible values of measurement of } A \]
\[ b_1, \ldots, b_N \quad \text{of } B \]
\[ c_1, \ldots, c_N \quad \text{of } C \]

\[ |1\rangle_A \ldots |N\rangle_A \]
\[ |1\rangle_B \ldots |N\rangle_B \]
\[ |1\rangle_C \ldots |N\rangle_C \]

Initial state vectors for \( a_1, a_N; b_1, b_N; c_1, c_N \)

\[ \langle A | 1 \rangle \ldots \langle A | N \rangle \]
\[ \langle B | 1 \rangle \ldots \langle B | N \rangle \]
\[ \langle C | 1 \rangle \ldots \langle C | N \rangle \]

Final state vectors for \( a_1, a_N; b_1, b_N; c_1, c_N \)

\[ \langle m | n \rangle_A = \langle m | n \rangle_B = \langle m | n \rangle_C = S_{mn} \]

\( A \langle m | n \rangle_B \) complex
A = \sum_{n=1}^{\infty} \ln >_{A} a_n <_{A^n} \\
B = \sum_{n=1}^{\infty} \ln >_{B} b_n <_{B^n} \\
C = \sum_{n=1}^{\infty} \ln >_{C} c_n <_{C^n} \\

What does \[ |11\rangle_{A} <_{A} (1 + 12) >_{A} <_{2}| mean? \]

(experiments select the first 2 states.)

\[ P_{ab} = \frac{< a | b \times b | a >}{< a | a > < b | b >} \rightarrow K | a b > | b a > \]

Same interpretation

\[ < a | a > = \text{mean value of measurement} \]

of A in state \[ |a\rangle \].

Relation between \[ A, \ln >_{A}, a_n \]

\[ A \ln >_{A} = \sum_{m=1}^{g} (m >_{A} a_{m} <_{A} m \ln >_{A} a_{m} <_{A}) = \]

\[ \sum_{m=1}^{g} (m >_{A} a_{m} <_{A} m \ln >_{A} a_{m} <_{A}) = \]

\[ a_{n} \ln >_{A} a_{n} \].
we also have

\[
\langle n | A | m \rangle = \sum_{n=0}^{\infty} \langle n | m \rangle_A a_n \langle m | n \rangle
\]

\[
= a_n \langle n | n \rangle
\]

we see that \( | n \rangle_A \) is an eigenvector of \( A \) with eigenvalue \( a_n \).

The statement \( \sum | n \rangle_A \langle n | = I \) means that any vector can be represented as a linear combination of the \( | n \rangle \)

\[
| b \rangle = \sum_{n=1}^{\infty} \langle n | b \rangle | n \rangle_A
\]

\[
\langle b | = \sum_{n=1}^{\infty} \langle b | n \rangle \langle n | A \rangle_A \langle m \rangle
\]

This means that the eigenvectors of \( A \) form an orthonormal basis \( (\langle n | m \rangle_A = \delta_{mn}) \) assuming \( (a_m \neq a_n \implies m \neq n) \)
Consider the eigenvalue equation

\[ B \langle n_B \rangle_B = b_n \langle n_B \rangle_B \]

we can make this into a matrix equation involving numbers using

\[ \mathbf{B} \mathbf{n}_B = b_n \mathbf{n}_B \]

\[ \sum_n \langle m | B | n \rangle_B \langle n | \mathbf{m} \rangle_B = b_n \langle m | n \rangle_B \]

multiply both sides of this by \( \langle e | \mathbf{m} \rangle_A \) on the left.

\[ \sum_n \langle e | B | n \rangle_B \langle n | \mathbf{m} \rangle_B = b_n \langle e | n \rangle_B \]

this can be written in matrix form

\[
\begin{pmatrix}
\langle n | B | n \rangle_A \\
\langle n | B | n \rangle_A \\
\langle n | B | n \rangle_A
\end{pmatrix}
\begin{pmatrix}
\langle e | n \rangle_B \\
\langle e | n \rangle_B \\
\langle e | n \rangle_B
\end{pmatrix} =
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\[ b_n \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A \langle \text{lin} \rangle \\ A \langle \text{lin} \rangle \\ A \langle \text{lin} \rangle \\ \vdots \end{pmatrix} \]

This has the form of a standard N \times N matrix eigenvalue problem

\[(B_{\text{em}} - b_n S_{\text{em}}) X_m = 0\]

This is a problem to determine both \( b_n \) and \( X_m \). To find the possible values of the measurements of \( B \),

\[
\det (B_{\text{em}} - \lambda S_{\text{em}}) = 0
\]

This is a degree \( N \) polynomial in \( \lambda \) with \( N \) roots \( b_1, \ldots, b_N \).

Solving for \( X_m \rightarrow A \langle \text{lin} \rangle \),

\[ 1\rangle_{13} = \sum b_i |m\rangle_{AA} A \langle \text{lin} \rangle \]

which gives a solution to the eigenvalue problem in the
basis of eigenvectors of $A$. If we use a different basis the $x_m$'s change but the eigenvector remain unchanged.

We considered operators like $| n \rangle_A \langle n |_{B} | m \rangle_{B}$, we can also consider products like $BC$ we can calculate this in the $A$-basis $B C | 1d \rangle = 1 \cdot B \cdot 1 \cdot C \cdot | 1d \rangle$

$$
2 \sum_{n} <n|_{A} <l|_{B} \langle 1 |_{A} <m|_{C} \langle n|_{A} <n|_{C}^{*} \langle 1d|_{A}.

$$

$$
2 \sum_{n} l_{n}^{A} B_{nm} C_{mn} d_{n}.

the $A$-components of the "vector" $BC|1d\rangle$ are obtained by multiplying the $A$-basis "matrix elements" of $B$ with the $A$-basis matrix elements of $C$ and applying the result to the $A$ components of the vector $|1d\rangle$. 

observable operators

\[ A = \sum \ln A^* a_n a_n^* \langle n \rangle \]

given A we define the adjoint operator \( A^\dagger \) by

\[ \langle a | A^\dagger b \rangle = \langle b | A^\dagger a \rangle^* \]

note for A above

\[ \langle a | A^\dagger b \rangle = (\langle b | \sum \ln A^* a_n a_n^* \langle n | a \rangle )^* \]

\[ = \frac{\mathcal{O}}{n!} \langle n_a | b \rangle a_n^* a_{n_a}^* \langle a | n_a \rangle \]

\[ = \frac{\mathcal{O}}{n!} \langle a | n_a \rangle a_{n_a}^* a_{n_a}^* \langle n_a | b \rangle \]

\[ = \langle a | (\sum \ln A^* a_n a_n^* \langle n_a | a \rangle ) b \rangle \]

we see that \( A = A^\dagger \) if the values of the measurements of A are real (i.e., the eigenvalues)
Observables operators in quantum mechanics are always self adjoint, $A = A^\dagger$ (Hermitean).

Consider the polynomials

$$
\prod_{m \neq n}^{\mu} (A - a_m) = \\
\prod_{m \neq n}^{\mu} (A - a_m) \cdot I = \\
\prod_{m \neq n}^{\mu} (A - a_m) \sum_{1R} |R \rangle \langle A_A | R \rangle = \\
\sum_{1R}^{\mu} \prod_{m \neq n}^{\mu} (a_R - a_m) |R \rangle \langle A_A | R \rangle
$$

Every term in this sum vanishes for $k \neq n$ because $k$ appears in the product $\prod_{m \neq n}^{\mu}$; what survives is

$$
|n \rangle \langle A_A | n \rangle = \prod_{m \neq n}^{\mu} (a_n - a_m)
$$

or

$$
|n \rangle \langle A_A | n \rangle = \prod_{m \neq n}^{\mu} \left( \frac{A - a_m}{a_n - a_m} \right)
$$
This shows that all of the elementary projections operators on eigenstates of $A$ are degree $N-1$ polynomials in $A$.

This is independent of the choice of basis.

It follows that

$$A = \sum_{n=1}^{N} a_n \prod_{m \neq n} \left( \frac{A-a_m}{a_n-a_m} \right)$$

$$A^2 = \sum_{n=1}^{N} a_n^2 \prod_{m \neq n} \left( \frac{A-a_m}{a_n-a_m} \right)$$

$$S(A) := \sum_{n=1}^{N} S(a_n) \prod_{m \neq n} \left( \frac{A-a_m}{a_n-a_m} \right)$$

$$A^{-1} = \sum_{n=1}^{N} \frac{1}{a_n} \prod_{m \neq n} \left( \frac{A-a_m}{a_n-a_m} \right)$$

We see that any function of $A$ is a degree $N-1$ polynomial in $A$.
we also note that if the roots of 
\( \det (\lambda I - A) \) are all distinct
the \( n \) von vanishing \( \langle n \rangle_{AA} \)
are defined

\[
\langle n \rangle_{AA} \langle m \rangle_{AA} = \]

\[
\prod \left( \frac{A - a_k}{\alpha_n - a_k} \right), \prod \left( \frac{A - a_k}{\alpha_m - a_k} \right)
\]

if \( m \neq n \) we can factor out

\[
\left[ \prod_{k=1}^{N} (A - a_k) \right]
\]

the statement that \( \prod (A - a_k) = 0 \)
is called the Cayley Hamilton theorem. - heuristic: prod

\[
\det (\lambda I - A) = \prod (\lambda - a_n) = 1
\]

\[
\prod (A - a_n) = \det (A - A) = 0
\]

since there are \( N \) \( \langle n \rangle_{AA} \) that
are independent; the \( \langle n \rangle_{AA} \) term
a basis.
When $a_m = a_n$ for $m \leq n$, it is still possible to find a basis of eigenvectors.