Lecture 41

\[ H = H_0 + V(t) \]
\[ V(t) = 0 \quad t \leq 0 \]
\[ H|\psi_n\rangle = E_n|\psi_n\rangle \quad \text{known} \]

\[ |\psi(0)\rangle = \sum |\psi_n\rangle C_n(0) \]
\[ |\psi(t)\rangle = \sum |\psi_n\rangle e^{-iE_n t / \hbar} C_n(t) \]

\[ \frac{d|\psi(t)\rangle}{dt} = -\frac{i}{\hbar} H|\psi(t)\rangle \]

This led to the following equation for the coefficients \( C_n(t) \)

\[ C_m(t) = C_m(0) - \frac{i}{\hbar} \int_{t-\infty}^{t} \sum_{n=0}^{\infty} \langle \psi_m V(t') \psi_n \rangle e^{-i\Omega_{mn} t'} C_n(t') dt' \]

\[ \Omega_{mn} = \frac{E_m - E_n}{\hbar} \]

Formally, this can be solved by iteration which generates a series in powers of the interaction \( V(t) \)

\[ C_m^{(0)}(t) = C_m(0) \]
\[ C_m^{(R)}(t) = C_m(0) - \frac{i}{\hbar} \int_{t-\infty}^{t} \sum_{n=0}^{\infty} \langle \psi_m V(t') \psi_n \rangle e^{-i\Omega_{mn} t'} C_n(t') dt' \]

\[ C_m(t) = \lim_{k \to \infty} C_m^{(R)}(t) \]
stopping this iteration after a finite number of steps gives a perturbative approximation to \( C_n(t) \)

if \( \|V\| < \omega \) and \( |t| < \omega \) it will be shown that this iteration converges.

**First order calculation**

\[
C_m(t) = C_m(t) = C_m(0) - \frac{i}{\hbar} \sum_{n=0}^{\infty} \int_0^t <\Psi_m(t')|I|\Psi_n(t')> e^{-i\omega_{mn} t} dt
\]

If we assume that the system is initially in the state \( "n" \) then

\[
C_m(0) = \delta_{mn}
\]

\[
C_m(t) = \delta_{mn} - \frac{i}{\hbar} \int_0^t <\Psi_m(t')|I|\Psi_n(t')> e^{-i\omega_{mn} t} dt
\]

\[
|\Psi(t)\rangle = |\Psi_n\rangle - \frac{i}{\hbar} \sum_{m} \langle \Psi_n | \Psi_m \rangle e^{-i\omega_{mn} t} \int_0^t <\Psi_m(t')|I|\Psi_n(t')> e^{-i\omega_{mn} t} dt
\]

Note that the evolution is norm preserving, but the corrections to the norm in perturbation theory do not appear until second order in \( V \).
\[ P = |\langle \psi(t) | \psi(t) \rangle |^2 = \\
\left| 1 - \frac{i}{\hbar} e^{iE_{nm}/\hbar} \int_0^t \langle \psi_m(t') | V(t') | \psi_n(t') \rangle \, dt' \right|^2 \\
\]

\[ P_m = |\langle \psi_m | \psi(t) \rangle |^2 = \\
\left| - \frac{i}{\hbar} e^{iE_{nm}/\hbar} \int_0^t \langle \psi_m(t') | V(t') | \psi_n(t') \rangle \, dt' \right|^2 \\
\]

We consider a useful special case where \( V(t) = V\Theta(t) - V \) is a time independent interaction.

Then the first form sol becomes:

\[ C_m(t) = C_m(0) - \frac{i}{\hbar} \sum_{n=0}^{\infty} \langle \psi_m | V | \psi_n \rangle \int_0^t e^{i\omega_{mn} t} C_n(0) \]

\[ \int_0^t e^{i\omega_{mn} t'} \, dt' = \frac{1}{i\omega_{mn}} \left( e^{i\omega_{mn} t} - 1 \right) \]

For the case where the sum has 1 term (i.e., we start in state \( n \))

\[ P_m = \frac{1}{\hbar^2} \left| \frac{1}{i\omega_{mn}} \left( e^{i\omega_{mn} t} - 1 \right) \right|^2 = \\
\frac{1}{\hbar^2 \omega_{mn}^2} \left( 2 - e^{-i\omega_{mn} t} - e^{i\omega_{mn} t} \right) \]
\[ P_m = \frac{4}{\hbar^2 \omega_{mn}} 2 (1 - \cos \omega_{mn} t) \langle \chi_m | \chi_n \rangle^2 \]

\[ = \frac{4}{\hbar^2 \omega_{mn}} \sin^2 \left( \frac{\omega_{mn} t}{2} \right) \langle \chi_m | \chi_n \rangle^2 \]

Note that \( \frac{\sin^2 (\omega t)}{\omega} \) is peaked at \( t = 0 \) - the slope increases with \( t \)

\[ \lim_{t \to 0} \frac{\sin^2 \lambda t}{\lambda^2 t} = \pi \]

\[ \lim_{t \to 0} \frac{\sin (\lambda_2 t)}{4 (\lambda_2)^2 t} = \frac{\pi}{4} \]

\[ \therefore \lim_{t \to 0} P_m(t) \to \frac{4}{\hbar^2} \cdot \frac{\pi}{4} t \delta \left( \frac{\omega_{mn}}{2} \right) \]

\[ = \frac{\pi t}{\hbar^2} \delta \left( \frac{E_m - E_n}{2\hbar} \right) \]

\[ = \frac{2\pi}{\hbar} \delta (E_m - E_n) t \]

(\text{This is actually a continuous distribution that approaches a delta function})
we write \[ \frac{S}{m} = \int dE_m \cdot \frac{dn}{dE_m} = \int dE \rho(E) \]
The quantity \( \rho(E) \) is called the density of states. (many different states can generally have the same energy)

\[ \frac{dW}{dt} = \text{Transition rate from } n \rightarrow \text{ all other states} \]
\[ = \frac{\rho}{m} \cdot \frac{dP_m}{dt} = \int \frac{2\pi}{\hbar} |\Psi_m \rangle \langle \Psi_n| \rho(E) dE \cdot \delta(E - E_n) \]

using the \( \delta \) function gives the following expression for the transition rate in first order perturbation theory. Integrating:

\[ \frac{dW}{dt} = \frac{2\pi}{\hbar} |\Psi_m \rangle \langle \Psi_n| \rho(E_m) \]

(there is an implied sum over all final states with the same energy as the initial state. This is called Fermi's Golden Rule.
This result is particularly useful for scattering.

**The Interaction Picture.**

\[ |\psi_s \rangle \text{ indep of time} \]

\[ |\psi_s \rangle \quad U(t) = e^{-iHt/\hbar} \]

In the interaction picture we remove the free interaction from \( \psi_s \):

\[ |\psi_I(t) \rangle = e^{iH_f t/\hbar} |\psi_s(t) \rangle \]

\[ i\hbar \frac{d|\psi_I(t) \rangle}{dt} = i\hbar \frac{d}{dt} \left( e^{iH_f t/\hbar} e^{-iH_f t/\hbar} \right) |\psi_I(t) \rangle \]

\[ = i\hbar \left\{ e^{iH_f t/\hbar} (iH_f t/\hbar - iH_f t/\hbar) e e e e \right\} \]

\[ = e^{iH_f t/\hbar} V(t) e^{-iH_f t/\hbar} |\psi_I(t) \rangle \]

We define

\[ V_I(t) \equiv e^{iH_f t/\hbar} V(t) e^{-iH_f t/\hbar} \]

\[ i\hbar \frac{d|\psi_I(t) \rangle}{dt} = V_I(t) |\psi_I(t) \rangle \]
In the interaction picture both operators and states have a time dependence. The inner products in all 3 pictures are identical.

\[ |\Psi_I(t)\rangle = |\Psi_I(0)\rangle \]

\[ |\Psi_I(t)\rangle = \frac{i}{\hbar} \int_0^t V(t') |\Psi_I(t')\rangle \, dt' \]

This is really an abstract representation of what we have done. It can be solved by iteration

\[ |\Psi_I^{(n)}(t)\rangle = |\Psi_I(0)\rangle - \frac{i}{\hbar} \int_0^t V_I(t') |\Psi_I^{(n-1)}(t')\rangle \, dt' \]

\[ |\Psi_I^0(t)\rangle = |\Psi_I(0)\rangle \]

It is useful to rewrite this series so it looks like an exponential series. This was first done by Dyson and is called the Dyson series in physics. They are called Volterra Integral equations.
consider the second order term

\[ |\psi_2(t)\rangle = |\psi_2(w)\rangle - \frac{i}{\hbar} \int_0^t V(t') |\psi_2(w)\rangle \, dt' + \left(\frac{-i}{\hbar}\right)^2 \int_0^t \int_0^{t'} V(t') V(t'') |\psi_2(w)\rangle \, dt' \, dt'' \]

note that in the second term \( t' > t'' \)

\[ \int_0^t dt' \int_0^{t'} dt'' \quad \int_0^{t''} dt'' \]

we can express the second order term as \( \frac{1}{2} \) the sum of these terms

\[ \frac{1}{2} \left(\frac{-i}{\hbar}\right)^2 \left[ \int_0^t dt' \int_0^{t'} dt'' V(t') V(t'') + \int_0^t dt' \int_0^{t''} dt' V(t') V(t'') \right] \]

next we relabel \( t' \rightarrow t'' \quad t'' \rightarrow t' \) in the second term

\[ \frac{1}{2} \left(\frac{-i}{\hbar}\right)^2 \left[ \int_0^t dt' \int_0^{t''} dt' V(t') V(t'') + \int_0^t dt' \int_0^{t''} dt' V(t') V(t'') \right] \]
The second integral is $0 \rightarrow t' \rightarrow t$ but in integrand changes.

We define

$$T(V_x(t)) V_x(t) = \Theta(t - t_i) V_x(t_i) V_x(t)$$

we call this the time ordered product

The second order term can be rewritten as

$$\frac{1}{2} \left( -\frac{1}{\hbar^n} \right) \int_0^t dt' \int_0^t dt'' T(V_x(t) V_x(t))$$

At 3rd order there are 3! possible operator orderings. A similar calculation gives

$$\frac{1}{3!} \left( -\frac{1}{\hbar^n} \right)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 T(V_x(t) V_x(t_1) V_x(t_2))$$

$$\vdots$$

$$\frac{1}{n!} \left( -\frac{1}{\hbar^n} \right)^n \int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} T(V_x(t_1) \cdots V_x(t_n))$$
The full solution becomes

\[ \Psi_t^{\pm} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{i}{\hbar})^n \int_0^t dt_1 \cdots \int_0^{t-n} dt_n \ T(V_{\pm}(t_1) \cdots V_{\pm}(t_n)) \mid \Psi_0 \rangle \]

\[ \equiv \text{Te}^{\phi \left(-\frac{i}{\hbar} \int_0^t dt + V_{\pm}(t)\right)} \mid \Psi_0 \rangle \]

This defines the **time ordered exponent** (Teo) It is used extensively in quantum field theory and many body quantum mechanics.

**Note**

\[ ||T(V_{\pm}(t_1) \cdots V_{\pm}(t_n))|| \leq ||V||^n \]

If \( V \) is a bounded operator (basically we get the same bound for any time ordering. The \( n \) term is bounded by

\[ \frac{1}{n!} \frac{t^n}{\hbar^n} ||V||^n \]

so the entire series is bounded by

\[ \frac{t^{||V||}}{n!} \]

\[ \epsilon < \infty \]
In many cases of interest $V$ is unbounded and we want to consider infinite time limits.

Example:

Consider a harmonic oscillator

$$H_0 = \frac{p^2}{2\hbar} + \frac{1}{2} m \omega^2 x^2$$

$|\tilde{n}\rangle$ eigenstate of $H_0$.

$$E_n = \hbar \omega (n_x + n_y + n_z + \frac{3}{2})$$

$$V(t) = \lambda (r^2)^2$$

If we start in the oscillator state $|\tilde{n}\rangle$, then to first order after time $t$,

$$P_m(t) = \frac{4}{\hbar \omega (m-\tilde{n}^2)} \sin^2 \left( \frac{\hbar \omega t}{2} \right) |\langle \tilde{n}(r^2)| \tilde{m}\rangle|^2$$

$$(r^2)^2 = (x^2 + y^2 + z^2)^2$$

These can be converted to creation and annihilation operators

$$\tilde{X} = \frac{1}{\sqrt{2}} \left( \tilde{a} + \tilde{a}^* \right) \sqrt{\frac{\hbar}{m\omega}}$$

$$\tilde{a} = \sqrt{\frac{\hbar \omega}{2m}} (x + i\hbar \rho/m\omega)$$

$$\tilde{a}^* = \sqrt{\frac{\hbar \omega}{2m}} (x - i\hbar \rho/m\omega)$$