operators that change basis
\[ \left\{ |n\rangle^n_a \right\}_{n=1}^\infty, \left\{ |n\rangle^n_b \right\}_{n=1}^\infty, \text{ 2 bases} \]

we define
\[ U_{ab} = \sum_{n=1}^\infty |n\rangle^n_a \langle n|_b. \]

using this definition
\[ U_{ab}|m\rangle_b = \sum_{n=1}^\infty |n\rangle^n_a \langle n|m\rangle_b = \sum_{n=1}^\infty |n\rangle^n_a \delta_{nm} = |m\rangle^n_a \]

we see that \( U_{ab} \) replaces the \( m^n \) basis state of \( B \) by the \( m^n \) basis state of \( A \).

As an example
\[ U_{yx} = |+\rangle^y_x \langle +| + |-\rangle^y_x \langle -| \]

has the effect of changing the \( x \)-basis to the \( y \) basis.
Let us compute $U_{ab}^t$

$\langle cl | U_{ab}^t | ld \rangle = \langle d l | U_{ab}^t | c \rangle^*$

$\sum_n \langle d l | n \rangle_{a b}^* \langle n | c \rangle^* =$

$\sum_n \langle c l | n \rangle_{b c} \langle n | l d \rangle =$

$\langle c l \left( \sum_n \langle n | b c \rangle \langle b c | n l \rangle \right) | l d \rangle$

or

$0 = \langle c l \left[ U_{ab}^t - \frac{2}{n} \langle n | b c \rangle \langle b c | n l \rangle \right] | l d \rangle$

this must be true for all $| c l \rangle, | l d \rangle$

$\Rightarrow U_{ab}^t = U_{ba} = \sum_n \langle b c | n \rangle \langle n c | l b \rangle$

Note

$U_{ab}^t U_{ab} = \sum_n \langle b c | n \rangle \langle n c | m \rangle \langle m b | l \rangle$

$= \sum_n \langle b c | n \rangle \delta_{n m} \langle m b | l \rangle$

$= \sum_n \langle b c | n \rangle \langle n l b | l \rangle$

$= I$

similarly
\[ \mathbf{U}_{ab} \mathbf{U}_{ab}^+ = \sum \mathbf{v}_n \langle \mathbf{v}_n | \mathbf{v}_n \rangle = \mathbf{I} \]

Change of basis operators have the property that the adjoint of the operator is the inverse of the operator. Operators with this property are called **unitary operators**.

Consider \( |a'\rangle = \mathbf{U} |a\rangle \) \( |b'\rangle = \mathbf{U} |b\rangle \)

\[ \langle a' | b' \rangle = \langle a' | \mathbf{U} | b \rangle = \langle b \mathbf{U}^+ | a' \rangle \]

\[ \langle b' | a' \rangle = \langle \mathbf{U} b \mathbf{U}^+ | a \rangle \]

\[ (\langle b' | a \rangle - \langle b \mathbf{U}^+ | a \rangle) = 0 \]

This means that

\[ \langle b' | a \rangle = \langle b \mathbf{U}^+ \]

and

\[ \langle b' | a' \rangle = \langle b \mathbf{U}^+ \mathbf{U} | a \rangle = \langle b \mathbf{I} | a \rangle = \langle b | a \rangle \]
This shows that unitary - or change of basis operators preserve the complex scalar product $\langle \alpha | \beta \rangle$.

(Note for real transformations in 3 dimensions the transformations that preserve the dot products of 3-vectors are rotations and reflections.)

Unitary operators generalize this concept to our complex inner product.

A consequence of this is

$P_{\alpha' \beta'} = |\langle \alpha' | \beta' \rangle|^2 = |\langle \alpha | \beta \rangle|^2 = P_{\alpha \beta}$.

Unitary transformations preserve probabilities in quantum mechanics. This means that the result of any experiment does not depend on the choice of basis.
If we consider eigenvalue problems
\[ A \langle n \rangle_a = a_n \langle n \rangle_a \]

If we change basis
\[ U A \langle n \rangle_a = a_n U \langle n \rangle_a \]
we define \[ \langle n' \rangle_a = U \langle n \rangle_a \]

\[ U A U^+ U \langle n \rangle_a = a_n \begin{align*} U \langle n \rangle_a \end{align*} \]
\[ \begin{align*} A' \quad \begin{align*} \langle n' \rangle_a \end{align*} \quad \begin{align*} \langle n' \rangle_a \end{align*} \end{align*} \]

If we define
\[ A' = U A U^+ \]
then \( A \) and \( A' \) have the same eigenvalues; the eigenvectors of \( A' \) are related to the eigenvectors of \( A \) by \[ U \langle n \rangle_a = \langle n' \rangle_a \]

\[ \langle d | A'^+ d \rangle = \langle d | A^+ c \rangle^* = \]
\[ \langle d | U A^+ U^+ c \rangle^* = \langle d | U^+ A^+ U^+ d \rangle \]
\[ \langle c | U A^+ U^+ c \rangle = \langle c | I U A^+ U^+ c \rangle \]
\[ \langle d | (A')^+ c \rangle \Rightarrow A' = (A')^+ \]
while $U^* = U^T$, $A = A^*$ correspond to different types of operators (change of basis vs observable) they are closely related.

Consider

$$U = (1-iA)(1+iA)^T =$$

$$= \sum \langle n | (1-ia_n) \langle n | m \rangle_{a} (1+ia_n)^T \rangle_{m} \langle m |$$

$$= \sum \langle n | \frac{1-ia_n}{1+ia_n} \langle n |$$

$$U^T = \sum \langle n | \frac{1+ia_n}{1-ia_n} \langle n |$$

$$U^T U = \sum \langle n | \left( \frac{1+ia_n}{1-ia_n} \frac{1-ia_n}{1+ia_n} \right) \langle n | = I$$

for every self adjoint operator $A$ there is a unitary operator $U = \frac{1-iA}{1+iA}$ with the same eigenvectors.
This relation can be inverted

\[
(1 + iA)U = 1 - iA
\]

\[
iA(U+1) = (1-U)
\]

\[
A = -i \, (U+1)^{-i}(1-U)
\]

clearly

\[
A^t = (1-U)^t \, (1+U)^{-t+i} \cdot i
\]

\[
= (1-U^t)(1+U)^{-t+i} \cdot i
\]

Note

\[
(1+U)^t(1+U)^{-t+i} = 1 \Rightarrow
\]

\[
(1+U^t)(1+U)^{-t} = 1
\]

\[
(1+U)^{-t} = (1+U^t)^{-t}
\]

\[
= (1-U^{-t})(1+U^{-t})^{-t}
\]

\[
= (U-1)U^t(1+U^{-t})^{-t}
\]

\[
= \underbrace{\text{the inverse of this}}_{(1+U^{-t})U = U+1 \Rightarrow}
\]

\[
= (U-1)(U+1)^{-t}
\]

\[
A^t = -i \, (1-U)(1+U)^{-t} = -i \, (1+U)^{-t}(1-U) = A
\]
\[ A = i \frac{u-1}{u+1} \quad u = \frac{1-iA}{1+iA} \]

This transformation is called a Cayley transform. It shows that there is a 1-1 correspondence between unitary and self adjoint operators.

\[ x \text{ This is not the only way to construct a unitary operator out of a self adjoint operator.} \]

0 we learned that we could write any function of an operator as a polynomial in that operator.

we cannot write \( S_x \) as a polynomial in \( S_x \).

\[ \frac{2}{\hbar} S_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \frac{2}{\hbar} S_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ S_x \frac{2}{\hbar} |+ \rangle_2 = |+ \rangle_2 \]

\[ S_x \frac{2}{\hbar} |- \rangle_2 = |+ \rangle_2 \]
for an arbitrary operator:

\[ B = \sum (|+\rangle_2 \langle +| + \langle +| - \leftrightarrow -1 + |+\rangle_2 \langle +| - \leftrightarrow +1 + |+\rangle_2 \langle +| - \leftrightarrow 1 + |+\rangle_2 \langle +| - \leftrightarrow -1)

= |+\rangle_2 \langle +| \left( \langle +| S_Z |+\rangle_2 \langle +| + \langle +| S_Z |-\rangle_2 \langle +| + \langle +| S_Z |+\rangle_2 \langle +| + \langle +| S_Z |-\rangle_2 \langle +| \right).

\frac{1}{n} S_X |+\rangle_2 \langle +| <+| S_Z |+\rangle_2 \langle +| + \frac{2}{n} S_X |-\rangle_2 \langle +| <+| S_Z |-\rangle_2 \langle +|.

Since \(|+\rangle_2 \langle +| - \leftrightarrow -1 \) are already polynomials in \( S_Z \) we see that any operator can be expressed as a polynomial in \( S_Z \) \( S_X \).

One property of \( S_X \) is that if we fix any eigenstate \( |\psi\rangle \) of \( S_X \), it has a probability \( \frac{1}{2} \) of being found in one of the eigenstates of \( S_Z \).
specifically if we prepare our system to be in the state \(|\uparrow_x\rangle\rangle\)
and measure \(S_x\) \(\frac{1}{2}\) of the measurements give \(|\uparrow_x\rangle\rangle\) and the other half give \(|\downarrow_x\rangle\rangle\). We get an identical result if we prepare the system in \(|\downarrow_x\rangle\rangle\).

By measuring \(S_x\), all information about the state of \(S_z\) is lost.

This is a property of complementary observables.

I discuss the general construction of complementary observables. We will use unitary transformations, but from what we learned about Cayley transforms there are observables with the same property.
Start with \( A = A^+ \)

\[ A |n\rangle_a = a_n |n\rangle_a, \quad n = 1, \ldots, N \]

define

\[ U |n\rangle_a = |n+1\rangle_a, \quad n < N \]
\[ U |N\rangle_a = |1\rangle_a \]
\[ U^2 |n\rangle_a = |n+2\rangle_a \]
\[ U^N |n\rangle_a = |N+n\rangle_a = |n\rangle_a \]
\[ (U^N - 1) |n\rangle_a = 0, \quad \text{for all } |n\rangle_a \]
\[ (U^N - 1) = 0 \]

The characteristic polynomial for this operator is

\[ (A^N - 1) = 0 \]

which has complex roots

\[ u_n = \exp\left(\frac{2\pi i n}{N}\right); \quad (u_n)^N = 1. \]
\[ \ln u < n \leq 1 = \prod_{m=n}^n \frac{U-U_m}{U_n-U_m} \]

\[ (U^n-1) = 0 = (U-u_n) \prod_{m=n}^n (U-U_m) \]

\[ \frac{U}{u_n} - 1 = \left( \frac{U}{u_n} - 1 \right) \sum_{m=0}^{n-1} \frac{(U}{u_n})^m = \frac{1}{u_n} (U-u_n) \sum_{m=0}^{n-1} \frac{U}{u_n})^m \]

\[ \prod_{m=n}^n (U-U_m) = \frac{1}{u_n} \sum_{m=0}^{n-1} \frac{U}{u_n})^m \]

\[ \therefore \ln u < n \leq 1 = (\text{const}) \times \sum_{m=0}^{n-1} \frac{U}{u_n})^m \]

To find the constant

\[ \ln u < n = \ln u < n \ln u < n \leq = \sum_{m=0}^{n-1} \frac{U}{u_n})^m = C \sum_{m=0}^{n+k} \frac{U}{u_n})^{m+k} \]

For each fixed value of \( k \), \( m \)

Increase \( N \) times - giving the full sum. \( k \) just shifts the starting point which does not matter because \( \left( \frac{U}{u_n} \right)^{m+k} \)

\[ l = CN \quad \Rightarrow \quad C = \frac{l}{N} \]
This gives

\[ \langle n | u \rangle u = \frac{1}{N} \sum_{m=0}^{\nu-1} (\frac{u}{u_n})^m \]

\[ \langle n | n \rangle_{u} \langle n | n \rangle_{u} = \frac{1}{N} \sum_{n=0}^{\nu-1} \frac{u^m}{u_n} | n \rangle \]

\[ = \frac{1}{N} \sum_{n=0}^{\nu-1} \langle n | m \rangle \frac{1}{u_n^m} \]

\[ = \frac{1}{N} \frac{1}{u_n^m} = \frac{1}{N} \]

we choose the phases so \( \langle n | n \rangle_{u} \) are real positive

\[ \langle n | n \rangle_{u} = u \langle n | n \rangle_{u} = \frac{1}{\sqrt{N}} \]

\[ \langle m | n \rangle_{u} \langle n | n \rangle_{u} = \]

\[ = \frac{1}{N} \sum_{k=1}^{\nu} k \frac{u^k}{u_n^k} | n \rangle \]

\[ = \frac{1}{N} \sum_{k=1}^{\nu} k \frac{1}{u_n^k} | k \rangle \]

\[ = \frac{1}{N} \frac{1}{u_n^m} = \langle m | u \rangle_{u} \cdot \frac{1}{u_n} \]

\[ \langle m | n \rangle_{u} = \langle n | m \rangle_{u}^* = \frac{1}{\sqrt{N}} e^{-\frac{2\pi |m| m}{N}} \]

\[ \langle m | m \rangle_{u} = \langle n | m \rangle_{u}^* = \frac{1}{\sqrt{N}} e^{-\frac{2\pi |m| m}{N}} \]
It follows that
\[ |\langle m | n \rangle_u|^2 = \frac{1}{N}. \]
for any choice of \( m, n \). This is just like the spin 1/2 case—any measurement of \( I \) observer completely eliminates all information about the state of the complementary operator (vector)

Define a new operator \( V \) by
\[ \langle n | V = \langle n | n+1 \]

Obviously
\[ \langle n | V^n = \langle n | \]
\[ (V^n - 1) = 0 \]
\[ \prod_{k=1}^n (V - u_k) = 0 \]
\[ 0 = (V - u_n) \prod_{k=1}^n (V - u_k) \]
\[ 0 = (V - u_n - 1) \prod_{k=1}^n (V - u_k) \]

As before we define
\[ 1 > \sum_{n=0}^{N-1} (V | u_n) \]
\[ \sum_{n=0}^{N-1} (V | u_n) = 1 \]
we choose the phase of $|\nu\rangle_v$ so:

$$\langle \nu \mid \nu \rangle_v \langle \nu \mid |\nu\rangle_u = \frac{1}{N} \sum_{m} \langle m \mid (\frac{V}{u})_{m} |\nu\rangle_u = \frac{1}{N}$$

$$\langle \nu \mid \nu \rangle_v = \langle \nu \mid |\nu\rangle_u = \frac{1}{\sqrt{N}}.$$

$$\langle \nu \mid \nu \rangle_v \langle \nu \mid |\nu\rangle_u = \frac{1}{N} \sum_{m} \langle m \mid (\frac{V}{u})_{m} |\nu\rangle_u = \frac{1}{N} \sum_{m} \frac{|u_{m}|^2}{u_{n}^2} = \frac{1}{N} \frac{|u_{m}|^2}{u_{n}^2}$$

$$\langle \nu \mid \nu \rangle_v = \frac{1}{\sqrt{N}} \sum_{m} u_{m}^* = \frac{1}{\sqrt{N}} e^{-2\pi i m n / N}$$

Note

$$|n\rangle_a = \sum_{m} \langle m \mid |\nu\rangle_u \langle m \mid n\rangle_a = \frac{1}{\sqrt{N}} \sum_{m} \langle m \mid |\nu\rangle_u \langle m \mid n\rangle_a = \frac{1}{\sqrt{N}} \sum_{m} \langle m \mid |\nu\rangle_u \langle m \mid n\rangle_a$$

$$|n\rangle_v = \sum_{m} \langle m \mid |\nu\rangle_u \langle m \mid n\rangle_v = \frac{1}{\sqrt{N}} \sum_{m} \langle m \mid |\nu\rangle_u \langle m \mid n\rangle_v$$

we see that the coefficients are identical so

$$|n\rangle_v = |n\rangle_a$$

which returns us back to the original set of eigenvectors.
$U$ and $V$ defined this way are complementary unitary operators—the corresponding self-adjoint operators are called complementary observable.

Note that

\[
C = \sum_{m/n} \langle n | \chi_m, m | \langle n | \chi_n \rangle \\
= \sum_{m/n} \langle n | \chi_n \rangle \langle n | \chi_m \rangle \\
\langle n | \chi_n \rangle = \langle n | \gamma_a = U^{n-m/n} \langle m | \gamma_a \rangle \\
C = \sum_{m/n} U^{n-m/n} \langle m | \gamma_a \rangle \langle m | \langle n | \chi_m \rangle \langle n | \chi_n \rangle \\
\text{polynomial in } V
\]

A general operator has the form

\[
C = \sum_{m/n}^{N-1} c_{mn} U^m V^n
\]
The coefficients $c_{mn}$ are related to matrix elements $a$ by

$$
\langle k|e\rangle_v = \sum_c c_{mn} \langle k|u^mv^n|e\rangle_v
$$

$$
= \sum_c c_{mn} e^{2\pi i n} \langle k|u^m|e\rangle_v
$$

$$
= \sum_c c_{mn} e^{2\pi i n} \langle k|l^m\rangle_v
$$

$$
= \sum_{k,e,m} c_{k-e,m} e^{2\pi i n}
$$

This can also be inverted to express $c_{mn}$ in terms of the matrix elements.