Lecture 7

Last time we were trying to generalize the relation between $S_x$ and $S_z$

The two main properties we wanted were

1. $\langle n | m \rangle_{S_x} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2}} \quad \forall \ n, m$,

2. $0 = \sum c_{mn} S_z^m S_x^n$

We started with an operator $A$ with eigenvalues and eigenvectors $\{a_n\}$ $|n\rangle_a$, $n=1\ldots N$.

We defined a unitary change of basis operator

$U |n\rangle_a = |n+1\rangle_a$

$U |1\rangle_a = |0\rangle_a$

$(U^{n-1}) = 0$, $(U^n - 1) = 0$ eigenvalues of $U$

$U$ are $u_n = e^{2\pi i n/N}$

Using

$|n\rangle_u = \prod_{m<n} \frac{U - U_m}{U_n - U_m}$
\[ \prod_{m} (U - u_m) = \prod_{m} (U - u_n) = U_n \left( \frac{u_n}{u_n - 1} \right)^{m+n} \]

\[ (U_n - 1) \]

\[ \left( \frac{U}{U_n} \right)^{n-1} = \left( \frac{u_n}{u_n - 1} \right) \sum_{k=0}^{n-1} \left( \frac{U}{U_n} \right)^{k} = \]

Comparing these equivalent expressions:

\[ \prod_{m} (U - u_m) = (\text{cons}) \times \sum_{k=0}^{n-1} \left( \frac{U}{U_n} \right)^{k} = (\text{cons})^n \ln \frac{u_n}{u_n} \]

Thus, we are led to:

\[ \ln \frac{u_n}{u_n} = c \sum_{k=0}^{n-1} \left( \frac{U}{U_n} \right)^{k} \]

To find \( c \) we use the identity:

\[ \ln \frac{u_n}{u_n} = \ln \frac{u_n}{u_n} \]

\[ c^2 \sum_{k=0}^{n-1} \left( \frac{U}{U_n} \right)^{k+1} = c \sum_{m=0}^{n-1} \left( \frac{U}{U_n} \right)^{m} \]

Note that for each fixed value of \( \ell \) we have:

\[ \left( \frac{U}{U_n} \right)^{\ell} \left( \frac{U}{U_n} \right)^{\ell+1} \ldots \left( \frac{U}{U_n} \right)^{\ell+n-1} \]

The next term in this sequence is:

\[ \left( \frac{U}{U_n} \right)^{\ell+n} = \left( \frac{U}{U_n} \right)^{\ell} \left( \frac{U}{U_n} \right)^{n} = \left( \frac{U}{U_n} \right)^{n} \]
so every power of \((u/u_n)\) appears in the k sum \(\sum\) any \(k\). Since there are \(N\) values \(k \leq \frac{N}{2}\) between \(k=0, ..., k=\frac{N}{2}\) we get

\[
C^2 \sum_{m=0}^{N-1} \left( \frac{u}{u_m} \right)^m = C \sum_{m=0}^{N-1} \left( \frac{u}{u_n} \right)^m
\]

or \(C = \frac{1}{N}\).

This gives

\[
\ln \frac{u}{u_n} = \frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^k
\]

Note that

\[
\left( \frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^k \right)^N = \frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^{-k}
\]

\[
\frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^k \left( \frac{u}{u_n} \right)^{-k} = \frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^{N-k}
\]

\[
\frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{u}{u_n} \right)^m = \frac{1}{N} \sum_{m=0}^{N-1} \left( \frac{u}{u_n} \right)^m
\]

because

\[
\left( \frac{u}{u_n} \right)^0 = 1 = \left( \frac{u}{u_n} \right)^N
\]
Consider
\[ \langle N | n \rangle_a \langle n | N \rangle_a = \]
\[ \frac{1}{N} \sum_{n=0}^{N-1} \langle N | n \rangle_a \langle n | N \rangle_a = \]
\[ \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{u_n} \langle N | n \rangle_a \langle n | N \rangle_a = \]
\[ \frac{1}{N} \cdot \langle N | N \rangle_a = \frac{1}{N} \]

We are free to choose the phases of \( |u\rangle_m \)
We choose them so
\[ \langle N | u \rangle_a = \langle u | N \rangle_a = \frac{1}{\sqrt{N}} \]

Next consider
\[ \langle N | u \rangle_a \langle n | m \rangle_a = \]
\[ \frac{1}{N} \cdot \sum_{k=0}^{N-1} \langle N | m+k \rangle_a \frac{1}{u_{n+k}} \] (one term survives)

\[ \frac{1}{N} \cdot \langle n | m \rangle_a = \frac{1}{N} \frac{1}{u_n^{N-k}} \]
\[ \quad = \frac{1}{N} u_n^{k-N} = \frac{1}{N} u_n^k \]

\[ \langle n | m \rangle_a = \langle m | n \rangle_a = \frac{1}{\sqrt{N}} e^{i \frac{2\pi nk}{N}} \]
It is clear from this equation that

\[ |\langle n| m \rangle_a| = \frac{1}{\sqrt{n}} \]

for any pair \( m, n \). This means that if we fix \( A \) in any \( N \) eigenstates \( \{n\} \), we measure a function of \( U \). The probability of getting any eigenvalue of \( \mathcal{S}(U) \) is \( 1/U \); independent of our initial or final vector. Measuring \( U \) destroys all information about \( A \).

It is also obvious that

\[ \langle n|_a \langle m |_a = U^{n-m} \langle l_m|_a \langle m |_a \]

\[ = U^{n-m} \prod_{k \neq m} \left( \frac{A-a_k}{A-a_{a_k}} \right) \]

\[ \sum \delta_{mn} \langle n|_a \langle m |_a = \sum \delta_{mn} \prod_{k \neq m} \left( \frac{A-a_k}{A-a_{a_k}} \right) \cdot I \cdot O \cdot I \]

which shows that any operator can be expressed as a polynomial in \( U \) and \( A \) of degree \( \leq N-1 \).
The operators $A$ and $U$ are called complementary observables.

We note that the relation is essentially symmetric.

If we define

$$
\langle n | V | n+1 \rangle_U = \chi_{n+1}^{n+1} \\
\langle n | U | n \rangle_U = \xi_n
$$

$$(V)^N - 1 = 0 \quad \bar{U}_n = U_n = e^{2\pi in/N}$$

$$\langle m | N \rangle^N = \frac{2^{N-1}}{N} \sum_{k=0}^{N-1} (\frac{\chi_k}{\chi_m})^k$$

$$\langle u | N | m \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \langle u | (\frac{\chi_k}{\chi_m})^k | N \rangle$$

We choose phases of $| m \rangle$ so

$$\langle u | n | m \rangle = \chi_{m}^{n} = \langle m | N | u \rangle = \frac{1}{\sqrt{N}}.$$

\[ \langle m | n \rangle_u = \langle m | n \rangle_v \frac{1}{\sqrt{\nu}} \]

\[ \langle m | \frac{1}{2} \left( \frac{v}{u} \right)^{k+m} | n \rangle_u = \left( \frac{1}{\sqrt{\nu}} \right)^{u-m} \frac{1}{\sqrt{\nu}} \]

\[ \langle m | \frac{1}{2} \left( \frac{v}{u} \right)^{k+m} | n \rangle_v = \frac{1}{\sqrt{\nu}} e^{\frac{2\pi i m}{n}} \]

\[ \langle m | l \rangle_u = \langle n | l \rangle_u \frac{1}{\sqrt{\nu}} e^{\frac{2\pi i m}{n}} \]

Finally note

\[ \langle n | l \rangle_u = \langle n | l \rangle_u \frac{1}{\sqrt{\nu}} e^{\frac{2\pi i m}{n}} \]

we see that

\[ \langle n | l \rangle_u = \langle n | l \rangle_v \]

we see that if \( A \) is complement of

\( B \), then \( B \) is complement of

\( A \).
It is clear that any operator can be expressed as a degree $N-1$ polynomial in $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{0} = \sum_{\nu} \langle n | \mathbf{0} | \nu \rangle \langle \nu | \mathbf{1} \rangle = \sum_{\nu} \langle \nu | \mathbf{0} \rangle \mathbf{1}$$

Since $\langle \nu | \mathbf{0} \rangle = \mathbf{U}^{n-m} \mathbf{1} \langle \nu | \mathbf{0} \rangle$

$$\langle \nu | \mathbf{0} \rangle \mathbf{1} = \mathbf{U}^{n-m} \mathbf{1}$$

$$= \prod_{k=m}^{N-m} \frac{\mathbf{V} - \mathbf{U}_k}{\mathbf{U}_k - \mathbf{U}_n}$$

which gives a different representation of the same operator

$$\mathbf{0} = \sum_{m=0}^{N-1} \mathbf{0}_{m\nu} \mathbf{U}_m \mathbf{V}_\nu$$

$$\langle n | \mathbf{0} | \nu \rangle = \sum_{k=m}^{N} \mathbf{0}_{k\nu} \langle n | \mathbf{U}^k \mathbf{U}_m \mathbf{V}_\nu \rangle$$

$$= \sum_{k=m}^{N} \mathbf{U}_m^k \mathbf{0}_{k\nu} \langle n | \mathbf{U}_m^k \mathbf{U}_m \mathbf{V}_\nu \rangle$$

$$= \sum_{k=m}^{N} \mathbf{U}_m^k \mathbf{0}_{k\nu} \langle n | \mathbf{U}_m \mathbf{V}_\nu \rangle$$

This expression can be inverted.
\[ 0 = \sum_m \langle m \mid \phi \rangle \langle m \mid \Omega \rangle \langle \Omega \mid m \rangle \]
\[ = 2 \sum \alpha^{m-p} \frac{1}{N} \left( \frac{\nu}{\nu_R} \right)^k \langle \Omega \mid m \rangle \langle \Omega \mid m \rangle \]
\[ = 2 \sum \frac{1}{N} \left( \frac{\nu}{\nu_R} \right)^k \langle \Omega \mid m \rangle \langle \Omega \mid m \rangle \]
\[ = 2 \sum \frac{1}{N} e^{-2\pi \nu \nu_R} \langle \Omega \mid m \rangle \langle \Omega \mid m \rangle \]
\[ = 2 \sum \frac{1}{N} e^{-2\pi \nu \nu_R} \langle \Omega \mid m \rangle \langle \Omega \mid m \rangle \]
\[ \tilde{O}_{mn} = \frac{2}{\nu} \frac{1}{\nu_R} e^{-2\pi \nu \nu_R} \langle \Omega \mid m \rangle \langle \Omega \mid m \rangle \]

one property that distinguishes complementary operators is \([A, B] \neq 0\)

\[ [A, B] = 0 \implies \]
\[ [P(A), P(B)] = 0 \implies \]
\[ [\langle n \rangle_{\alpha} \langle n \mid \langle m \rangle_{\alpha} \langle m \rangle_{\beta}] = 0 \]
\[ \langle n \rangle_{\alpha} \langle n \mid \langle m \rangle_{\alpha} \langle m \rangle_{\beta} = \langle n \rangle_{\alpha} \langle n \mid \langle m \rangle_{\alpha} \langle m \rangle_{\beta} \]

\[ A \left( \langle n \rangle_{\alpha} \langle n \rangle_{\beta} \langle m \rangle_{\alpha} \langle m \rangle_{\beta} \right) = a_n \langle n \rangle_{\alpha} \langle n \rangle_{\beta} \langle m \rangle_{\alpha} \langle m \rangle_{\beta} \]
\[ B \left( \langle m \rangle_{\alpha} \langle m \rangle_{\beta} \langle n \rangle_{\alpha} \langle n \rangle_{\beta} \right) = b_m \langle m \rangle_{\alpha} \langle m \rangle_{\beta} \langle n \rangle_{\alpha} \langle n \rangle_{\beta} \]
This means that

\[ |n\rangle_a |m\rangle_b = L_a m_b \]

is an eigenstate of \( A, B \) with eigenvalues \( a_n \) for \( A \) and \( b_m \) for \( B \)

\[ I = \sum_{m,n} |n\rangle_a m_b \rangle \langle m,n| \]

\([A,B]=0 \] are called commuting observables. It is always possible to find simultaneous eigenstates of both \( A \) and \( B \).

You can think of \( A, B \) are being measurements that refine the state of a quantum system.

Consider an operator \( A = \Pi \)

\[ A \rightarrow |n\rangle_a = |n\rangle_v \quad \text{on} \quad u_v \]

\[ U \quad |n\rangle_u = |n\rangle_n \quad \text{on} \quad u_n \]

\[ \mathcal{V} \quad |n\rangle_v = |m\rangle_u \quad \text{on} \quad u_v \]

\[ |\langle n|m| \rangle = \frac{1}{\sqrt{N}} \]
Assume $N = N_1 N_2$

$6 = 2 \cdot 3$

$11 \rightarrow 12 \rightarrow 13 \rightarrow 14 \rightarrow 15 \rightarrow 16$

$111 \rightarrow 112 \rightarrow 113 \rightarrow 121 \rightarrow 122 \rightarrow 123$

$U_1 \langle m n \rangle_v = \langle m+1 n \rangle_v$

$U_2 \langle m n \rangle_v = \langle m n+1 \rangle_v$

$\langle m n \rangle_u \langle n m \rangle_u = \frac{1}{N_4} \frac{1}{N_2} \sum \left( \frac{U_1}{X_m} \right)^{R_1} \left( \frac{U_2}{X_n} \right)^{R_2}$

$\langle m n \rangle_u V_1 = \langle m+1 n \rangle_u$

$\langle m n \rangle_u V_2 = \langle m n+1 \rangle_u$

$[U_1, U_2] = [V_1, V_2] = [U_1, V_1] = [U_2, V_2] = 0$

$U_1 V_1 = V_1 U_1 e^{-2 \pi i / \omega_1}$

$U_2 V_2 = V_2 U_2 e^{-2 \pi i / \omega_2}$

$O = \delta_{m, m_1, n, n_1} V_1^{m_1} V_2^{n_1} U_1^{m} U_2^{n}$

This can be repeated until $|\psi\rangle$ is decomposed into states of different commuting observables with prime numbers of eigenvalues.