Lecture 8

Continuous quantum numbers

\[ N = \# \text{ eigenvalues} \quad (\text{large prime}) \]

\[ X_n = e^{\frac{2\pi i n}{N}} \quad n = -\left(\frac{N-1}{2}\right) \ldots 0 \ldots \left(\frac{N-1}{2}\right) \]

\[ V^{k} U^{k} = U^{k} V^{k} e^{\frac{2\pi i k}{N}} \]

to take the continuum limit

\[ U = e^{i\epsilon q}, \quad V = e^{i\epsilon p} \]

Here \( p, q \) are Hermitian operators

\[ U^\dagger = U^{-1} \Rightarrow q^\dagger = -q \quad p^\dagger = +p \]

Let \[ \frac{2\pi}{N} = e^i \quad l \epsilon = q^\dagger \quad m \epsilon = p^\dagger \quad l, m \epsilon = \left(-\frac{N-1}{2}, \frac{N-1}{2}\right) \]

\[ l \epsilon, m \epsilon \rightarrow \left(\frac{2\pi/l\epsilon - 1}{2}\right), \left(\frac{2\pi/m\epsilon - 1}{2}\right) \times \epsilon = -\left(\frac{\pi - \epsilon}{2}\right) + \left(\frac{\pi - \epsilon}{2}\right) \]

as \( \epsilon \rightarrow 0 \quad N \rightarrow \infty \quad p, q \rightarrow -\infty, \infty \)

\[ V^l = e^{i(l\epsilon) p} = e^{i\epsilon p} \quad \{ \text{primed quantities} \} \]

\[ U^m = e^{i(m\epsilon) q} = e^{i\epsilon q} \quad \{ \text{are numbers} \} \]

\[ V^l U^m = U^m V^l e^{\frac{2\pi i k}{N}} \]

\[ i\epsilon q, l\epsilon q, i\epsilon p, l\epsilon p \text{ each } \epsilon = e \ldots e \ldots e \ldots e \]

\[ e 

\]
Recap

1. States of a quantum system are represented by vectors in complex vector space with a complex inner product (Hilbert space). \( \langle a | b \rangle = \langle b | a \rangle^* \)

2. \( |a|^2 \) represents the probability that a system in state \( a \) will be measured to be in \( b \rangle \)

This assumes \( \langle a | c \rangle = 1 \) \( \langle b | b \rangle = 1 \)

If not,

\[
    P_{ab} = \frac{\langle a | b \rangle \langle b | a \rangle}{\langle a | c \rangle \langle c | a \rangle}
\]

3. Observables are represented by self-adjoint linear operators on the Hilbert space.

\( \circ \) Observables have a complete set of orthogonal eigenvectors

\( \circ \) Observables have real eigenvalues
\[
\langle a_1 A_1 b \rangle^* = \langle b_1 A_1^\dagger c \rangle \quad \text{defines } A_1^+ \\
\text{self adjoint } \iff A = A_1^+ \\
\text{or self adjoint operators always have properties (a) (b).} \\
\]

4. expectation values of operator
\[
\langle b_1 A_1 b \rangle = \sum_n a_n \langle b_1 c \rangle \langle c_1 b \rangle \\
= \sum_n K_{ab} k_x^2 \\
\text{probability}
\]

5. ensemble averages
\[
\rho = \sum P_n \langle b_n \rangle \langle b_1 \rangle \\
\text{Tr}\rho = \sum P_n = 1, \quad 0 \leq P_n \leq \rho \\
\langle A \rangle_\rho = \sum P_n \langle b_n A_1 b \rangle \\
= \sum P_n \sum a_{nm} \langle b_n b_m \rangle \\
\]
Choosing a basis reduces all quantum mechanical calculation to matrix algebra. 

2. Change of basis is done by unitary operation $U$, $UU^*=U^*U=I$

$$<a|b>^* <c|d> = \text{Tr}(\rho A)$$

have basis independent values. 

3. Every operator $A$ has a complementary operator $C$

$$K_{on} C_{mn} = \frac{1}{\sqrt{N}}$$

4. Every operator is a polynomial in $U$, $UV$ complementary. 

5. Commuting observables can be simultaneously diagonalized. 

6. All quantum systems can be decomposed into irreducible building blocks.
\[ U(p') = e^{i q' p} \]
\[ V(q') = e^{i q' p} \]
\[ V(q') U(p') = U(p') V(q') e^{i q' p} \]

*Write this as*

\[ U(p') = V(q') U(p') V(q') e^{i q' p} \]
\[ V(q') = U(p') V(q') U^+(p') e^{i q' p} \]

\[ p' q = p' (V(q') q V(q') + q') \]
\[ q' p = q' (U(p') p U^+(p') + p') \]

*These identities are only true up to phase.*

\[ V^+(q') q V(q) = q - q' \]
\[ V(q') q V^+(q') = q + q' \]
\[ U(p') p U^+(p') = p - p' \]
\[ |q'\rangle = |q'\rangle |q\rangle \]

\[ q |V(q')\rangle = |V(q')\rangle (V^\dagger(q') q |V(q')\rangle |q\rangle = |V(q') (q-q') |q\rangle = (q'-q') |V(q')\rangle \]

\[ V(q') |q\rangle = (q'-q') \]

\[ V^\dagger(q') |q\rangle = (q'+q') \]

\[ P |p\rangle = |p\rangle |p\rangle \]

\[ P U(p') |p\rangle = U(p') U^\dagger(p) P U(p) |p\rangle = U(p') (p+p') |p\rangle = (p'+p') U(p') |p\rangle \]

\[ U(p') |p\rangle = |p\rangle |p\rangle \]

\[ U^\dagger(p') |p\rangle = |p\rangle |p\rangle \]

As before the $UV$ operators shift the eigenvalues.
completeness

\[ I = \sum_{n} n < n' \]
\[ = \int n > d n < n' \]
\[ = \int n(n') > \frac{d n}{d n'} d n < n' (n') \]

\[ \frac{d n}{d n'} = \frac{1}{\epsilon l} = \frac{1}{\epsilon} \]
\[ = \int n > \frac{1}{\sqrt[4]{\epsilon}} d n \frac{1}{\sqrt[4]{\epsilon}} < n' \]

\[ | n(n') > = n > \frac{1}{\sqrt[4]{\epsilon}} \]

\[ I = \int_{-\infty}^{\infty} | n > d q < n' | \]

Similarly

\[ I = \int_{-\infty}^{\infty} | p > d p < p' | \]

If we calculate

\[ \langle A|A > = \langle A|\Sigma|A > = \]
\[ \sum_{-\infty}^{\infty} \langle A | q > d q < q | A > = \]
\[ \int_{-\infty}^{\infty} \langle A | p > d p < p | A >. \]
\[ \langle q_1 A \rangle = A(q) \text{ is called the wave function of } |A\rangle \text{ in the } q \text{ representation} \]

\[ dp = K q_1 A K'^2 dq' = \]

probability of measuring \( q \) within \( dq' \) of \( q' \) if the system is prepared in state \( A \).

\[ K q_1 A K = \text{probability density} \]

\[ \langle q_1 A \rangle = \text{wave function - probability amplitude} \]

The requirement

\[ \int_{-\infty}^{\infty} K q_1 A K'^2 dq' = \lim_{|q| \to \infty} K q_1 A K'^2 \to 0 \]

This implements the requirement that large values of \( q \) are irrelevant - allowing us to avoid problems with periodic boundary conditions.
change of basis

$$\int_{-\infty}^{\infty} \langle p' | q' \rangle \, dq' \langle q' | p' \rangle$$

Recall

$$\langle q' | p' \rangle = \frac{1}{\sqrt{2\pi}} \mathcal{E}$$

$$\langle q' | p' \rangle = \frac{1}{\sqrt{2\pi}} \mathcal{E} e^{-i p' \cdot q'}$$

$$\langle q' | 1 p' \rangle = \frac{1}{\sqrt{2\pi}} \mathcal{E} e^{-i p' \cdot q'}$$

$$\langle 1 p' | q' \rangle = \frac{1}{\sqrt{2\pi}} \mathcal{E} e^{i p' \cdot q'}$$

$$\langle 1 p' | 1 p' \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{E} e^{i q'' \cdot (p' - p''')} \, dq''$$

This quantity makes no sense

$$\langle 1 p' \rangle = \int \langle 1 p' | 1 p' \rangle$$

$$\langle 1 p' \rangle = \int \langle 1 p' | 1 p' \rangle$$

$$\langle 1 p' \rangle = \int \mathcal{E} A(p'') \frac{1}{2\pi} \mathcal{E} e^{i q'' \cdot (p'' - p')}$$

If we do the $p$ integrals first, the $q$ integral makes sense.
we are led to the normalization of continuous eigenstates
\[ \langle q' \mid q'' \rangle = \mathcal{S}(q' - q'') \]
\[ \langle p' \mid p'' \rangle = \mathcal{S}(p' - p'') \]
does this mean?

\[
\begin{cases}
\mathcal{S}(q) = e^{-q^2} \\
\mathcal{G}(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i p (q - q')} g(q') \, dp dq'
\end{cases}
\]

since any reasonable function can be approximated by a linear combination of Gaussians — we see that this hold generally.

Let \( \mathcal{S}_n(q) \) be any functions with the property:

1. \( \int \mathcal{S}_n(q) = 1 \) for any \( n \)
2. \( \mathcal{S}_n(q) = 0 \) for \( q \in [-\frac{\pi}{2}, \frac{\pi}{2}] \).

Then

\[ \int \mathcal{S}_n(q - q') g(q') = \lim_{n \to \infty} \int \mathcal{S}_n(q - q') g(q') \]

as long as \( g(q') \) is nice — almost const where \( \mathcal{S}_n \to 0 \).
Note \[ \int_{-\infty}^{\infty} e^{-q^2} dq = \sqrt{\pi} \]

To interpret
\[ \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{iq''(p''-p'')} dq'' = \langle p'' | p' \rangle \]

\[ \langle p'' | p' \rangle = \int \langle p'' | p' \rangle dp' \langle p' | p' \rangle = \]

\[ = \frac{1}{2\pi} \int e^{iq''(p''-p'')} \langle p' | p' \rangle dp'dq'. \]

This means that if we integrate
\[ F(p''-p') = \frac{i}{2\pi} \int e^{iq''(p''-p'')} dq'' \]

\[ \int F(p''-p') \Psi(p') dp' = \Psi(p) \]

While \( F \) is not really a function, it is a perfectly good linear operator.

\[ F \Psi'' = \Psi''(p'') \]

\[ F [ \Psi_i + \alpha \Psi_j] = \Psi_i''(p'') + \alpha \Psi_j''(p'') \]

\[ \delta(p''-p') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iq''(p''-p')} dq'' \]

\[ \int \delta(p''-p') \Psi(p') dp' = \Psi(p) \]
we define
\[ \frac{dS(q-q')}{dq'} = \lim \int \frac{d^q s_{ff}}{dq}, s_{f}(q-q')g(q') = \]
\[ - \lim \int s_{f}(q-q') \frac{dg}{dq} = \int s_{f}(q-q') \left(-\frac{d^q g}{dq}, g(q')\right) = \]
\[ - \frac{dg}{dq} \quad etc. \]

there are many related limits

consider
\[ e^{i p \cdot q} e^{-i p \cdot q'} = q - q'. \]
\[ \langle q | p | \Psi \rangle = -i \frac{2}{dq}, \quad \langle q | e^{i p \cdot q'} | \Psi \rangle = \]
\[ = -i \frac{2}{dq}, \quad \langle q | V(q') | \Psi \rangle = \]
\[ = -i \frac{2}{dq}, \quad \langle q+q' | 14 \rangle = \]
\[ = -i \frac{2}{dq}, \quad \langle q | 14 \rangle. \]

thus in the q representation
\[ P = -i \frac{2}{dq}. \]
\[ [P, q^{-1}] = -i \]
consistent with our previous case.