Lecture 8

complementary observables

$U, V$ unitary

$u \langle m | n \rangle_v = \frac{1}{\sqrt{N}} e^{\frac{2\pi i m n}{N}} \quad \langle m | n \rangle_v^2 = \frac{1}{N}

U \langle m | n \rangle_u = e^{2\pi i m n/N} \langle m | n \rangle_u

V \langle m | n \rangle_v = e^{2\pi i m n/N} \langle m | n \rangle_v

U \langle n | n \rangle_u = \langle n | n \rangle_u

V \langle n | n \rangle_v = \langle n | n \rangle_v

\begin{align*}
UV &= u \sum_{n} \langle n | e^{2\pi i m n/N} | n \rangle_v \\
&= \sum_{n} \langle n+1 | e^{2\pi i m n/N} | n \rangle_v

VU &= \sum_{n} \langle n | e^{2\pi i m n/N} | n \rangle_u \\
&= \sum_{n} \langle n | e^{2\pi i m n/N} | n-1 \rangle
\quad m = n-1 \\
&= \sum_{n} \langle m+n | e^{2\pi i m n/N} | m \rangle e^{2\pi i n/N} \\
&= \sum_{n} \langle m+n | e^{2\pi i m n/N} | m \rangle e^{2\pi i n/N} \\
&= \sum_{n} \langle m+n | e^{2\pi i m n/N} | m \rangle e^{2\pi i n/N}
\end{align*}

\begin{align*}
\text{let} \quad m \rightarrow n \\
VU &= u \langle n | e^{2\pi i n/N} \\
&= \langle n | u | n \rangle e^{2\pi i n/N}
\end{align*}
representations of operators

$$\Theta = \sum_{n} \langle n | \Theta | m \rangle \langle m |$$

$$= \sum_{mn} \tilde{\Theta}_{mn} V^m U^n$$

To relate \( \tilde{\Theta}_{mn} \) \( \langle n | \Theta | m \rangle \)

$$\langle n | \Theta | m \rangle = \sum_{RE} \tilde{\Theta}_{RE} \langle n | V^R U^E | m \rangle$$

$$= \sum_{RE} \langle n | V^R U^{E \times n-m} | m \rangle \tilde{\Theta}_{RE}$$

$$= \sum_{k} \langle n | V^R | m \rangle e^{2\pi i n k} \tilde{\Theta}_{R_{n-m}}$$

The inverse can be obtained

$$\Theta = \sum_{n} \langle n | \Theta | m \rangle \langle m |$$

$$\sum_{n} \langle n | \Theta | m \rangle \langle m | V^{-} e^{2\pi i m n}$$

let \( k = -m + n = n - m \)

$$\sum_{n} \langle n | U^k | n \rangle \langle n | \Theta | n - k \rangle$$

$$= \sum_{n} \frac{(V - e^{2\pi i m n})}{(e^{2\pi i m n} - e^{2\pi i m n})} U^k \langle n | \Theta | n - k \rangle$$

The coefficient of \( V^m U^k \) is \( \tilde{\Theta}_{mk} \)
commuting observables

\[ [A, B] = 0 \implies AB = BA \]

using this it is easy to show

\[ [P(A), Q(B)] = 0 \] for any polynomials in \( A \) \& \( B \)

\[ |n\rangle_a \langle n| \] \& \[ |m\rangle_b \langle m| \]

are both polynomials in \( A \) \& \( B \)

this means that

\[ (|n\rangle_a \langle n| |m\rangle_b \langle m|) = (|m\rangle_b \langle m|) |n\rangle_a \langle n| \]

\( A |n\rangle_a \langle n| |m\rangle_b \langle m| = a_n |n\rangle_a \langle n| |m\rangle_b \langle m| \)

\( B |n\rangle_a \langle n| |m\rangle_b \langle m| = b_m |m\rangle_b \langle m| |n\rangle_a \langle n| \)

\[ = b_m |m\rangle_b \langle m| |n\rangle_a \langle n| \]

\[ = b_m |n\rangle_a \langle n| |m\rangle_b \langle m| \]

If we apply ( ) to any \( |c\rangle \)

\[ (|n\rangle_a \langle n| \langle m|_b \langle m| \) |c\rangle \]
it the result is not 0 then it is an eigenstate of both A and B with eigenvalues $a_n$ and $b_m$.

Physically, this means that if we measure A to be in the state $a_n$ and B to be in the state $b_m$ - the system remains in the combined state.

It is useful to write

\[(|n\rangle_{\alpha^n, b} \langle m|_{\beta^m}) = |n_m\rangle \langle n |_{\alpha^n, \beta^m}\]

where

\[A |n_m\rangle = a_n |n_m\rangle\]
\[B |n_m\rangle = b_m |n_m\rangle\]

\[T = \sum_{n,m} \langle n |_{\alpha^n, \beta^m} \langle n |_{\alpha^n, \beta^m}\]

where these are useful is when A has 2 identical eigenvalues; they may be in different eigenstates of B.
\{ a_n b_m \} \text{ when non-zero these are distinct eigenstates of } A \text{ with the same } a_n \text{ but different } B \text{ eigenvalues.}

By choosing a maximal number of independent commuting observables we can identify single states by specifying the eigenvalue of each commuting observable.

These define a minimal set of measurements needed to uniquely characterize the state of a system.

Let \( A \) have \( N = M \cdot K \) eigenvectors.

\[ \Rightarrow \text{ instead of labeling the eigenstates } \]
\[ 11 \rangle_a, 10 \rangle_a, \ldots, 1K \rangle_a \]
\[ (111 \rangle_a, 112 \rangle_a, \ldots, 11K \rangle_a) \]
\[ (1M1 \rangle_a, 1MK \rangle_a) \]
\[|11\rangle_a = |111\rangle_a\]
\[|12\rangle_a = |112\rangle_a\]
\[|n\rangle_a = |m+n\rangle_a\]

Instead of defining a single shift, we define \(U_1, V_1, U_2, V_2\)

\[U_1 |mn\rangle_a = |m+1, n\rangle_a\quad U_1^{m-1} = 0\quad U_{1n} = e^{\frac{2\pi \text{i} n}{m}}\]
\[U_2 |mn\rangle_a = |m, n+1\rangle_a\quad U_2^{k-1} = 0\quad U_{2n} = e^{\frac{2\pi \text{i} n}{k}}\]

Complementary states

\[|m \cdot n\rangle_a \rightarrow |mn\rangle_u\]

\[\langle m \cdot n | \text{ke}_u | \rangle^2 = \frac{1}{m \cdot k} = \frac{1}{15}\]

\(V_1, V_2\)

\[\langle m \cdot n | V_1 = \langle m+1 | n \rangle\]
\[\langle m \cdot n | V_2 = \langle m | n+1 \rangle\]

We have

\[\left\{U_1, U_2\right\} = \left\{V_1, V_2\right\} = \left\{U_1, V_2\right\} = \left\{U_2, V_1\right\} = 0\]

\[V_1 U_1 = U_1 V_1 e^{\frac{2\pi \text{i} m}{m}}\]
\[V_2 U_2 = U_2 V_2 e^{\frac{2\pi \text{i} k}{k}}\]
and operator becomes a polynomial in $V, V_j, U_1, U_2$. Note if $n = (n_1, n_2)$

$$V^n = V_1^{n_1} V_2^{n_2}$$

etc.

This process can be repeated until $N$ is factored into prime factors,

$$|n\rangle \rightarrow |n_{i_1}, \ldots, n_{i_k}\rangle$$

each eigenvalue is associated with a complementary pair $V_1, U_1$ of operators.

The reason for discussing commuting observables is that they define a complete measurement that uniquely determines the state of the system.
Some things we measure in quantum mechanics like the position or momentum of a particle have continuous eigenvalues.

We approximate these by using a limit of a large number of discrete eigenvalues.

Let $N$ be very large.

$$X_n = e^{2\pi i n/N}, \quad n = 0, \ldots, N-1$$

$$V^n U^n = U^n V^n e^{2\pi i k / N}$$

To take a continuous limit we let

$$U = e^{i e q}, \quad V = e^{i e p}$$

where $q$ and $p$ are Hermitian operators

$$U^* = U^{-1} \Rightarrow q = q^*$$

$$V^* = V^{-1} \Rightarrow p = p^*$$

Let

$$\frac{2\pi}{N} = e^2 \quad e \in \mathbb{Q}^*, \quad e = p^* \quad m = \left(-\frac{N-1}{2}, \frac{N-1}{2}\right)$$

(this prime $N$, $N$ is odd, so this is symmetric).
\[ V^e = (e^{i Cp})^p = e^{i (Cp)p} = e^{i p^r} \]
\[ U^m = (e^{i Cq})^m = e^{i (Cq)m} = e^{i p^q} \]
\[ V^e U^m = U^m V^e e^{\frac{2 \pi i m_0}{m}} \]

\[ e^{i p^r} e^{i p^q} e^{i p^r} e^{i p^q} = e^{i p^r} \]
\[ e^{i p^q} e^{i p^r} e^{i p^q} e^{i p^r} = e^{i p^q} \]

We can write this as
\[ e^{i p^r} e^{i p^q} e^{i p^r} e^{i p^q} = e^{i p^r (q + q')} \]
\[ e^{i p^q} e^{i p^r} e^{i p^q} e^{i p^r} = e^{i p^q (p + p')} \]

Using
\[ UF(A)U^\dagger = f(UAU^\dagger) \]

Which is an immediate consequence of \( f(A) \) being a polynomial in \( A \).
If we limit the allowed range of $p, q$
p'q' to be finite (but large) we
never have to worry about reaching
$2\pi$

\[
e^{i(p'q' + iq')} = (p + p')
\]
\[
e^{i(p'q' - iq')} = (q + q')
\]

if we differentiate with respect to 
p'q'

\[
i[p, q] = 1
\]
\[
-i[q, p] = 1
\]

\[
[p, q] = -i
\]

Normally, the argument of an
exponent is dimensionless. When
they have dimensions they must
be multiplied by a constant
that cancels the dimension.

$pq = \frac{px}{\hbar}$ for example
Recall
\[ \frac{2\pi}{\lambda} = e^2 \quad m \epsilon = p \quad n \epsilon = q \]
\[ I = \sum_{\nu_{/2}}^{\nu_1} \ln > \langle n \mid = \int_{-\nu_{/2}}^{\nu_1} \ln > \langle n \mid d\nu < n \mid \]
\[ = \int \ln > \frac{d\nu}{\nu} < n \mid d\nu \]
\[ = \frac{\pi}{e^2} \epsilon \]
\[ = \frac{\pi}{e^2} \epsilon \int \ln > \frac{i\nu}{\nu_e} < n \mid d\nu \]
\[ = \frac{2\pi}{\epsilon} \epsilon \]

We define \[ \mid q \rangle = \int \frac{i\nu}{\nu_e} d\nu < n \mid \]
Let \[ \epsilon \to 0 \]
\[ I \to \int_{-\infty}^{\infty} \mid q \rangle d\nu < n \mid \]

Note
\[ \langle q \mid p \rangle = \frac{\epsilon}{2\mu} \langle n_g \mid m_p \rangle = \frac{\epsilon}{\mu} \epsilon^{2n\mu m_p} \]
\[ = \frac{\epsilon}{2\mu} \cdot \frac{\mu}{\epsilon} \cdot e^{i\mu q} = \frac{2n\mu m_p}{\epsilon^2 \mu} e^{i\mu q}. \]
\[ \langle p' i p'' \rangle = \int \langle p' i q \rangle dq \langle q i p'' \rangle \]
\[ = \frac{1}{2\pi} \int e^{i q (p'' - p')} dq \]
\[ = s(p - p') \]