1. Consider a particle of mass $m$ in a one-dimensional square well of depth $V_0$ and width $L$ centered at the origin.
   a. What is the classical momentum of the particle?
   b. Where are the classical turning points?
   c. What is the structure of the WKB wave function in the classically forbidden region?
   d. Write down an equation for the WKB eigenvalues?
   e. Solve for the eigenvalues?

2. Consider a variational wave function of the form $\psi(r, \theta, \phi) = N e^{-ar} Y_0^0(\theta, \phi)$
   a. Find the normalization constant for the wave function.
   b. Let $H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{r}$ be the Hamiltonian for a one-electron atom.
      Calculate the expectation value of this Hamiltonian in this state.
   c. Find a value of $a$ that gives a variational bound on the ground state energy.
   d. Is this an upper or lower bound?
   e. How could this be modified to get a variational bound on the lowest energy state with orbital angular momentum $l = 1$ state.

3. Consider a system of three particles in states $\phi_1(r_1)$, $\phi_2(r_2)$ and $\phi_3(r_3)$
   Assume that each single particle wave function is normalized to unity and localized in distinct regions of space.
   a. Assume that the three particles are identical fermions. What is the form of the unit normalized wave function for this three particle system?
   b. Assume that the three particles are identical bosons. What is the form of the unit normalized wave function for this three particle system?
   c. Compare the probability distributions
      \[ |\Psi(r_1, r_2, r_3)|^2 dr_1 dr_2 dr_3 \]
      for the wave functions in parts a.) and b.)

4. Consider a particle of mass $m$ scatters from an infinitely high spherical barrier of radius $R$. 
5. Consider a Hamiltonian of the form

\[ H = H_0 + \lambda V \]

where

\[ H_0 = a I + b \sigma_z \quad V = c \sigma_x \]

and \( I \) is the 2x2 identity, the \( \sigma_i \) are Pauli matrices and \( a, b, \) and \( c \) are constants.

a. What are the eigenvalues and eigenvectors of \( H_0 \)?
b. What is the first order correction to the eigenvalues due to \( V \)?
c. What is the first order correction to the eigenvectors due to \( V \)?
d. What is the second order correction to the eigenvalues due to \( V \)?

6. Electrons bound to a molecule can be considered to be in a potential well. For small displacement about the equilibrium position of the well the restoring force approximately linear. Thus for the lowest energy eigenstates the restoring force behaves like a 3 dimensional harmonic oscillator with frequency \( \omega_0 \). If this molecule interacts with a weak oscillating electric field

\[ V(t) = -eE \sin(\omega t) \]

a. Find the probability as a function of time the this electric field causes a transition from the ground state to the first excited state?
b. How should the frequency be of the field be chosen to maximize this probability?

\[ x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \]

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2} \]

\[ j_0(x) = \frac{\sin(x)}{x} \quad n_0(x) = \frac{\cos(x)}{x} \]

\[ \sin(A) \cos(B) - \sin(B) \cos(A) = \sin(A - B) \]
Final Exam Solutions

average 157.6 / 220
standard deviation 34.7

10. \( P_{el} = \sqrt{2m(E + V_0)} \quad \text{E < 0 V_0 + E > 0 in the classically allowed region} \)

1b. \( \pm \frac{L}{2} \)

1c. \( \psi(x) = N \frac{1}{\sqrt{2mE_1}} e^{-\frac{1}{\hbar} \sqrt{2mE_1} x} \)

1d. \( \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{2m(E + V_0)} = \pi \hbar (n + \frac{1}{2}) = \sqrt{2m(V_0 + E)} L \)

1e. solve \( \ln E \quad E = -V_0 + \frac{\hbar^2 k^2}{2mL^2} (n + \frac{1}{2})^2 \)

2a. Note \( \int_{0}^{\alpha} r^2 e^{-2\alpha r} dr = (\frac{1}{2 \alpha})(\frac{1}{2 \alpha})^n = (\frac{1}{2})^n \frac{N!}{\alpha^n} \)

\( l = N^2 \int_{0}^{\alpha} r^2 e^{-2\alpha r} dr = N^2 \left( \frac{1}{2} \right)^3 \frac{2}{\alpha^3} = \frac{N^2}{4\alpha^3} \)

\( N = 2\alpha^3 \)

2b. \( 4\alpha^3 \int_{0}^{\alpha} \left( -\frac{k^2}{2m} (\alpha^2 r^2 - 2\alpha r) - Ze^2 r \right) e^{-2\alpha r} dr = 4\alpha^3 \left( -\frac{k^2}{2m} \left( \frac{1}{2} \right)^2 \frac{2}{\alpha^3} - 2\alpha \left( \frac{1}{2} \right)^3 \frac{1}{\alpha^2} \right) = -\frac{k^2}{2m} (\alpha^2 - 2\alpha^2) - Ze^2 \alpha = \frac{k^2}{2m} 0^2 - Ze^2 \alpha = \psi_{11} \psi_{11} \)

2c. \( \frac{d}{d\alpha} \psi_{11} = 0 = \frac{k^2}{2m} 2\alpha - Ze^2 \quad \alpha = \frac{Ze^2 m}{\hbar^2} \)

\( \psi_{11} \psi_{11} \) \( l_0 = \frac{k^2}{2m} \frac{Z^2 e^4 m^3}{\hbar^4} - Ze^2 \frac{Ze^2 m}{\hbar^2} = -\frac{Z^2 e^4 m}{\hbar^2} \)

2d. least upper bound on \( E \)

2e. for \( l = 1 \) it is sufficient to include \( -\frac{k^1}{2m} \left( \frac{\hat{L}^2}{\hbar^2 r^2} \right) = -\frac{2}{2mr^2} \) in \( H \)
3a) Fermions \[ \Psi(r_1, r_2, r_3) = \]
\[ \frac{1}{\sqrt{6}} \left( \phi_1(r_1) \phi_2(r_2) \phi_3(r_3) + \phi_1(r_2) \phi_2(r_3) \phi_3(r_1) + \phi_1(r_3) \phi_2(r_1) \phi_3(r_2) \right) \\
- \phi_1(r_2) \phi_3(r_1) \phi_2(r_3) - \phi_1(r_3) \phi_3(r_2) \phi_2(r_1) - \phi_1(r_1) \phi_2(r_3) \phi_3(r_2) \]

3b) Bosons \[ \Psi(r_1, r_2, r_3) \]
\[ \frac{1}{\sqrt{6}} \left( \phi_1(r_1) \phi_2(r_2) \phi_3(r_3) + \phi_1(r_2) \phi_2(r_3) \phi_3(r_1) + \phi_1(r_3) \phi_2(r_1) \phi_3(r_2) \right) \\
+ \phi_1(r_2) \phi_3(r_1) \phi_2(r_3) + \phi_1(r_3) \phi_3(r_2) \phi_2(r_1) + \phi_1(r_1) \phi_2(r_3) \phi_3(r_2) \]

3c) because \[ \phi_i(r_3) \phi_k(r_3) = 0 \text{ for } i \neq k \]
\[ d\mathbf{P} = |\phi_1(r_1) \phi_2(r_2) \phi_3(r_3)|^2 dr_1 dr_2 dr_3 = \]
\[ |\phi_1(r_1) \phi_2(r_2) \phi_3(r_3)|^2 dr_1 dr_2 dr_3 \]

the result is the same for bosons and fermions.

4a) \[ 0 < r < R \]
\[ a j_0 \left( \frac{kr}{n} \right) + b n_0 \left( \frac{kr}{n} \right) \]
\[ r > R \]

4b) \[ \Psi(R) = 0 \]

4c) \[ 4b \Rightarrow \Psi(r) \rightarrow N \left( j_0 \left( \frac{kr}{n} \right) n_0 \left( \frac{kr}{n} \right) - n_0 \left( \frac{kr}{n} \right) j_0 \left( \frac{kr}{n} \right) \right) \]
\[ = \frac{N n_0}{kr} \left( \sin \left( \frac{kr}{n} \right) \cos \left( \frac{kr}{n} \right) - \cos \left( \frac{kr}{n} \right) \sin \left( \frac{kr}{n} \right) \right) \]
\[ = \frac{N n_0}{kr} \sin \left( \frac{kr}{n} - \frac{kr}{n} \right) = 0 \]

4d) \[ G = \frac{4\pi k^2}{r^2} \sin^2 \theta_e = \frac{4\pi k^2}{r^2} \sin^2 \left( \frac{kr}{n} \right) \]
(5a) \[ E_\pm = a \pm b \left( \frac{1}{\sqrt{2}} \right) \text{ for } a \pm b \left( \frac{1}{\sqrt{2}} \right) \text{ for } a \pm b \]

(5b) \[ \begin{align*}
\langle +1 \mid c^0 \mid 1^+ \rangle &= 0 \\
\langle -1 \mid c^0 \mid 1^- \rangle &= 0 \\
\end{align*} \]

\[ E^1_\pm = 0 \]

(5c) \[ \begin{align*}
\begin{align*}
1^+ \langle +1 \mid c^0 \mid 1^+ \rangle &= 1^+ \frac{c}{2b} &= \frac{c}{2b} \left( \frac{1}{\sqrt{2}} \right) \\
1^+ \langle +1 \mid c^0 \mid 1^- \rangle &= 1^+ \frac{c}{2b} &= -\frac{c}{2b} \left( \frac{1}{\sqrt{2}} \right)
\end{align*}
\end{align*} \]

(5d) \[ \langle \pm 1 \mid c^0 \mid 1^+ \rangle \langle \pm 1 \mid c^0 \mid 1^- \rangle = \pm \frac{c^2}{2b} \]

(5e) \[ C(001) = 0 - \frac{i}{\hbar} \int_0^t e^{i\hbar \omega_0 \left( \frac{\xi}{\hbar} \right)} \langle 001 \mid -eEz \mid 1000 \rangle = \]

\[ = -\frac{i}{\hbar} (-eE) \sqrt{2\pi} \int_0^t e^{i(\omega - \omega)t} dt \langle 001 \mid (a_x + a_y) \mid 1000 \rangle \]

\[ = \frac{eE}{\sqrt{2\pi} \hbar \omega_0} e^{i\omega t} \sin \left( \frac{\omega t}{2} \right) \]

P = |C(001)|^2 = \frac{2e^2E^2}{m\hbar \omega_0} \sin^2 \left( \frac{\omega t}{2} \right)

(6b) \[ \omega = \omega_0 \text{ maximizes } P. \]