1. Calculate the differential cross section (in terms of the transition operator matrix elements) for two-body scattering in the laboratory frame where particle 1 of mass $m_1$ is initially at rest, particle 2 of mass $m_2$ is initially moving with momentum $p = p_2$ and the angular distribution of particle 2 is measured. (Hint - you need to find the initial relative velocity in this frame and integrate over all variables that are not measured.)

2. Assume that an electron scatters off of a potential due to a spherically symmetric electric charge density, $-e\rho(r)$. Find the scattering amplitude and differential cross section in the Born approximation. Show how these are related to the Fourier of this charge distribution.

3. Consider a three-dimensional scattering problem for two particles of mass $m$ scattering with a potential

$$\langle P', k' | V | P, k \rangle = -\lambda \delta(P' - P) \frac{1}{a^2 + k'^2} \frac{1}{a^2 + k^2}$$

Solve the Lippmann Schwinger equation exactly to find

$$\langle k' | T \left( \frac{k^2}{2\mu} + i\epsilon \right) | k \rangle$$

4. For the potential of problem 3 find the scattering amplitude $F(k', k)$ and the differential cross section in the center of mass frame.

5. For the potential of problem 3 calculate the total cross section using the optical theorem.

6. For the potential of problem 3 the Born approximation can be obtained from the exact solution by keeping only the term in the scattering amplitude that is linear in the coupling constant $\lambda$. Compute the differential cross section in the Born approximation and compare the result to the exact cross section.
\( ds = \frac{(2\pi)^4}{1 V_F} \left| \frac{K \tilde{p}_1 \tilde{p}_2 | T(E+i\epsilon) | P_1 P_2 > |^2}{\text{momentum conserving \& factored out}} \right| \delta^4 (E_1' + E_2' - E_1 - E_2) \delta^3 (P_1' + P_2' - P_1 - P_2) \ d^3 P_1' \ d^3 P_2' \)

In this case:

0. \( \tilde{P}_1 = 0 \quad \tilde{P}_2 = \tilde{P} \quad E_1 = 0 \quad E_2 = \frac{P^2}{2m_2} \)

\( V_F = \frac{P^2}{m_1} - \frac{P^2}{m_2} = \frac{\tilde{P}}{m_2} \)

0. Integrating over \( \tilde{P}_1' \) eliminates the \( S \) function in momentum

\( \tilde{P}_1' = \tilde{P} - \tilde{P}_2' \)

0. The energy \( S \) function becomes

\( S \left( \frac{P_1'^2}{2m_1} + \frac{(P_2 - P_2')^2}{2m_1} - \frac{P_1^2}{2m_2} \right) = \)

\( S \left( \frac{P_2'^2}{2m_2} + \frac{P_2^2}{2m_1} - \frac{\tilde{P}_2' \tilde{P}}{m_1} - \frac{P_1^2}{2m_2} + \frac{P_1^2}{2m_1} \right) \)

0. We need

\( \int_0^\infty S \left( \frac{P_1'^2}{2m_1} + \frac{P_2^2}{2m_1} - \frac{P_2' \tilde{P} \cos \theta}{m_1} - \frac{P_1^2}{2m_2} + \frac{P_1^2}{2m_1} \right) P_2'' \ dp_2'' = \)

\( \frac{P_1'^2}{m_2 + \frac{P_1''^2}{m_1} - \frac{\tilde{P} \cos \theta}{m_1} \left| \frac{P_2''}{m_2} - \frac{P_2' \tilde{P} \cos \theta}{m_1} - \frac{P_1^2}{2m_2} + \frac{P_1^2}{2m_1} \right| \)

where \( P_1' = 0 \) of

\( \frac{P_2'^2}{2m_2} - \frac{P_2' \tilde{P} \cos \theta}{m_1} - \frac{P_1^2}{2} \left( \frac{m_2 - m_1}{m_2 m_1} \right) = 0 \)
This gives
\[
\Delta = \frac{2\pi}{1\pi^4} < \hat{P}'_2 \cdot \hat{P}'_2' \mid T(E+ic) \mid 0 \cdot \hat{P} >^2 \frac{P}{2\pi} \frac{m^2_2}{P_2' \cdot P_2' + P_{\text{cosec}}^2} \quad d\Omega_2'
\]

where
\[
0 = P_2'^2 - \frac{2\mu}{m_1} P \cos \theta P_2' - \frac{P^2}{m_1} \frac{m_2 - m_1}{m_1 + m_2}
\]
\[
P_2' = \frac{2\mu}{m_1} P \cos \theta \pm \sqrt{\frac{4\mu^2 P^2 \cos^2 \theta}{m_1^2} - 4P^2 \frac{(m_2 - m_1)}{m_1 + m_2}}
\]
\[
= \frac{\mu}{m_1} \cos \theta \pm \frac{\mu}{m_1} \sqrt{\cos^2 \theta - \frac{m_2 - m_1}{m_1 + m_2}}
\]
\[
= \frac{\mu}{m_1} \left( \cos \theta \pm \sqrt{\cos^2 \theta - \frac{(m_2 - m_1)}{m_2}} \right)
\]
\[
= \frac{\mu}{m_1} \left( \cos \theta \pm \sqrt{\cos^2 \theta + \frac{m_1}{m_2}} \right)
\]
\[
= \frac{\mu}{m_1 + m_2} \left( \cos \theta + \sqrt{\frac{m_1}{m_2} - \sin^2 \theta} \right)
\]

(for \( m_1 > m_2 \), the positive root is physical

\[
P_2' = \frac{\mu}{m_1 + m_2} \left( \cos \theta + \sqrt{\frac{m_1}{m_2} - \sin^2 \theta} \right)
\]

\( m_2 > m_1 \) \( \sin \theta \) m = \sqrt{\frac{m_1}{m_2}} \) is the largest possible scattering angle
\[ u(r) = \int -\frac{e\rho(r')}{|r-r'|} \, d^3r' \quad v = -e\nabla u(r) \]

In the Born Approximation

\[ \langle \tilde{k}' | T(E+ie) | \tilde{k} \rangle = \langle \tilde{k}' | V \tilde{k} \rangle = \]

\[ \int \langle \tilde{k}' | \tilde{r} \rangle \, eV(r) \, d^3r \, \langle \tilde{r} | \tilde{k} \rangle = \]

\[ -i(k' - \tilde{k}) \cdot \tilde{r}/\hbar \cdot \int \frac{1}{(2\pi \hbar)^3} \, e^{-i(k' - \tilde{k}) \cdot \tilde{r}/\hbar} \, (\frac{\rho(r')}{|r-r'|}) \, d^3r \, d^3r' \]

change variables \( r'' = r-r' \) so

\[ d^3r \, d^3r' = d^3r'' \, d^3r' \quad \tilde{r} = \tilde{r}'' + \tilde{r}' \]

\[ -\frac{e^2}{(2\pi \hbar)^2} \, e^{-i(k-\tilde{k}) \cdot (\tilde{r}'' + \tilde{r}')/\hbar} \cdot \int \frac{\rho(r')}{|r''|} \, d^3r' \, d^3r'' = \]

\[ \int \langle \tilde{k}' | r'' \rangle \, \frac{\rho}{\hbar} \, \langle r'' | \tilde{k} \rangle \cdot \int e^{-i(k-\tilde{k}) \cdot \tilde{r}''/\hbar} \, d^3r'' \]

\[ \langle \tilde{k}' | V \phi \tilde{k} \rangle = \langle \tilde{k}' | \phi \tilde{k} \rangle \cdot (2\pi \hbar^3 \, \tilde{\rho} (i\tilde{k}' - \tilde{k}')) \]

Fourier transform
Let $U = \frac{\lambda u <u|u> I}{\lambda u <u|u>}$

$I = \text{identity in } \mathbb{P}$

$\langle k|U \rangle = \frac{1}{\alpha + k^2}$

$T(z) = I \times (-\lambda u <u| - \lambda u <u| \frac{1}{z-\mu} ) T(z)$

This means $T(z)$ must have the form

$T(z) = \gamma |u><u| + I$

$\gamma |u><u| + I = (-\lambda u <u| - \lambda u <u| \frac{1}{z-\mu} ) |u><u| I$

$\gamma = -\lambda - \lambda |u| \frac{1}{z-\mu} |u| > \gamma$

$\gamma (z) = \frac{-\lambda}{1 + \lambda |u| (z-\mu)^{-1} |u|}$

To solve this we must compute $|u| (z-\mu)^{-1} |u| = $

$\int \frac{1}{\alpha + k^2} \frac{d^2 k}{z - \lambda u^2 + i \epsilon} \frac{1}{\alpha + k^2} =$

$\frac{-1}{2 \alpha} \int \frac{d^2 k}{d \alpha} \left( \int_0^\infty \frac{k^2 dk}{(\alpha + k^2)(k^2 - u^2 - k^2) + i \epsilon} \right) =$

$(-)^\frac{1}{2} \left( \frac{1}{2 \alpha} \right) 4 \pi u^2 \int \left( \int_0^\infty \frac{k^2 dk}{(k+i \alpha)(k-i \alpha)(k-k_0+i \epsilon)(k-k_0+i \epsilon)} \right)$

Close in the upper half plane
\[
= - \frac{2\pi i}{a} \left\{ \frac{-a^2}{2ia} - \frac{1}{a^2 - k_o^2} + \frac{k_o^2}{k_o^2 + a^2} \cdot \frac{1}{2k_o} \right\} \\
= - \frac{4\pi^2 i}{a} \frac{1}{da} \left\{ \frac{k_o - i a}{2a^2 + k_o^2} \right\} \\
= - \frac{2\pi^2 i}{a} \frac{1}{da} \left\{ \frac{1}{k_o + ia} \right\} = - \frac{2\pi^2 i}{a} \frac{(-1)^i}{(k_o + ia)^2} \\
= - \frac{2\pi^2 i}{a} \frac{1}{(k_o + ia)^2}
\]

Thus, given

\[
\tilde{\chi}(z) = \tilde{\chi}\left( \frac{k_o}{2a} + i e \right) = \frac{-\lambda}{1 - \frac{2\pi^2 i}{a} \frac{1}{(k_o + ia)^2}}
\]

\[
\langle k' | T(z) | k \rangle = \frac{1}{k^2 + a^2} \frac{-\lambda}{1 - \frac{2\pi^2 i}{a} \frac{1}{(k_o + ia)^2}} \frac{1}{k^2 + a^2}
\]

\[\triangleright \]

\[
F(k'k) = - (2\pi)^2 \mu \frac{1}{\lambda} \langle k' T \left( \frac{k_o}{2a} \right) | k \rangle = \\
- \frac{(2\pi)^2 \mu \frac{1}{\lambda}}{(k^2 + a^2)^2} \frac{1}{1 - \frac{2\pi^2 i}{a} \frac{1}{(k_o + ia)^2}} \\
\frac{d\sigma}{d\Omega} = \frac{(2\pi)^9 \mu^2 \hbar^2 \lambda^4}{(k^2 + a^2)^4} \frac{(k_o^2 + a^2)^2}{[(k_o + ia)^2 - \frac{2\pi^2 i}{a}]^2}
\]
\[ G_t = \frac{4\pi^2}{k} \text{Im} F(kh) \]

In this case, \( F \) is independent of direction so we only need to compute the imaginary part of \( F \)

\[
F = \frac{2\pi l^2 \mu k}{(a^2 + k_0^2)^2} \left( \frac{1}{1 - \frac{2\pi^2 l^2 \mu}{a(k_0 + ia)^2}} \right)^2
\]

\[
= \frac{2\pi l^2 \mu k}{(a^2 + k_0^2)^2} \left( \frac{k_0^2 - a^2 + 2ika}{k_0^2 - a^2 + 2ika - 2\pi^2 l^2 \mu/a} \right)
\]

\[
= \frac{2\pi l^2 \mu k}{(a^2 + k_0^2)^2} \left( \frac{k_0^2 - a^2 + 2ika}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2} \right)
\]

We can read off the imaginary part

\[
\text{Im} F = \frac{2\pi l^2 \mu k}{(a^2 + k_0^2)^2} \frac{(2k_0a)(2\pi^2 l^2 \mu/a)}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2}
\]

\[
= \frac{(2\pi)^4 \mu^2 \kappa^2 k_0}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2} \left( \frac{1}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2} \right)
\]

\[
G = \frac{(4\pi)^2 (2\pi)^2 \mu^2 k^4}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2} \left( \frac{1}{(a^2 + k_0^2)^2} \right)
\]

This can be compared to the result of integration:

\[
G = \int \frac{d\theta}{2\pi} dR = 4\pi \times \frac{d\theta}{2\pi} \quad \left( \frac{d\theta}{2\pi} \text{ is independent of } \theta \right)
\]

\[
= 4\pi \left( \frac{2\pi^2 l^2 \mu^2 k^4}{(a^2 + k_0^2)^2} \right) \left( \frac{1}{(k_0 + ia)^2 - 2\pi^2 l^2 \mu/a} \right)^2
\]

\[
= 4\pi \left( \frac{2\pi^2 l^2 \mu^2 k^4}{(a^2 + k_0^2)^2} \right) \left( \frac{1}{(k_0^2 - a^2 - 2\pi^2 l^2 \mu/a)^2 + 4k_0^2a^2} \right)
\]

which are identical.
(c) expanding the solution to problem 3 in powers of $\lambda$

$$<\mathbf{k}^* \mathbf{T}(\mathbf{z}) \mathbf{k}> = \frac{1}{a^2 + k^2} \left( -\lambda \right) \sum_{n=0}^{\infty} \left[ \left( \frac{2\pi^2 \mathbf{u}^2}{a} \right)^n \frac{1}{(k_n n^2 a)^2} \right] \frac{1}{\sqrt{e^2 + a^2}}$$

The term in brackets is

$$1 + \lambda \frac{2\pi^2 \mathbf{u}}{a} \frac{1}{(k_n n^2 a)^2} + \lambda^2 \left( \frac{2\pi^2 \mathbf{u}}{a} \right)^2 \frac{1}{(k_n n^2 a)^4} + \cdots$$

This approaches 1 as $\lambda \to 0$ or $k_n n^2 a$ get large.