Symmetry

1. Wigner's Theorem
2. Discrete symmetries
3. Galilean boosts
4. Gauge symmetries

**Wigner's Theorem**

Consider a correspondence between states

$$|\Psi\rangle \rightarrow |\Psi'\rangle$$

that satisfies

$$K|\Psi\rangle |\Psi\rangle^2 = K|\Psi'\rangle |\Psi'\rangle^2$$

for all $|\Psi\rangle, |\Psi'\rangle$. Then we can choose phases so the correspondence is unitary and linear or antunitary and antilinear.

The beauty of this theorem is that there are only these two possibilities.
unitary means
\[ \psi' = U \psi \]
\[ \phi' = U \phi \]
\[ \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle. \]

Antunitary means
\[ \psi' = A \psi \]
\[ \phi' = A \phi \]
\[ \langle \psi' | \phi' \rangle = \langle \phi' | \psi' \rangle = \langle \psi | \phi \rangle^* \]

The proof that I give follows the proof in S. Weinberg, Quantum Theory of Fields I, Page 91.

Let \( \{ \psi_m \} \) be an orthonormal basis.

Let \( \{ \phi_m \} \) be the corresponding set of vectors assumed in Wigner's theorem.
Then

1) $\langle \psi_m | \psi_n \rangle = \langle \psi_m | \psi_n \rangle = \delta_{mn}$

$\langle \psi_m | \psi_m \rangle = \langle \psi_m | \psi_m \rangle = 1$

$\langle \psi_m | \psi_n \rangle = 0$

Also if there is a vector $|k'\rangle$ orthogonal to all $\{ |\psi_m \rangle \}$

$\langle k | \psi_m \rangle = \langle k' | \psi_m \rangle = 0 \Rightarrow$

$\langle k | \psi_m \rangle = 0$ all $m$

This contradicts the completeness assumption about $\{ |\psi_m \rangle \}$

It follows that

$\{ |\psi_m \rangle \}$ orthonormal basis

$\{ |\psi'_m \rangle \}$ orthonormal basis

(2) Next we consider

$|s_R \rangle = \frac{1}{\sqrt{2}} |\psi_1 \rangle + \frac{1}{\sqrt{2}} |\psi_2 \rangle$

$\langle s_R | s_R \rangle = 1$
\[ 13' \rangle = 2 \ c_r \ 14'_r \rangle \]

Since
\[ \langle 5'_r | 13'_r \rangle = K |5'_r \rangle |5'_r \rangle = |5'_r \rangle \]
\[ 13'_r \rangle = \langle 5'_r | 13'_r \rangle = 1 |8_0 \rangle \ 1/2 + 8_0 \ 1/2 = 1 \ c_r \ 1 \]

Thus,
\[ 13'_r \rangle = \frac{1}{\sqrt{2}} \ e^{i \phi_r} 14'_r \rangle + \frac{1}{\sqrt{2}} \ e^{i \phi_k} 14'_k \rangle \]
\[ e^{-i \phi_r} 13'_r \rangle = \frac{1}{\sqrt{2}} \ 14'_k \rangle + \frac{1}{\sqrt{2}} \ e^{i (\phi_k - \phi_r)} 14'_k \rangle \]

For each \[ 13'_r \rangle \ 14'_k \rangle \] we are free to choose the phases. The phases can be redefined so
\[ 13'_r \rangle = \frac{1}{\sqrt{2}} \ 14'_k \rangle + \frac{1}{\sqrt{2}} \ 14'_2 \rangle \]

We do this ... for every \[ k \]

Now consider an arbitrary vector.
\[ 1n \rangle = 2 \ c_n \ 14_n \rangle \]
\[ 1n' \rangle = 2 \ c'_n \ 14'_n \rangle \]
compute

\[ |\langle n_i | \hat{s}_R \rangle| = |\langle n_i | \hat{s}_K \rangle| \]

\[ |C_i'|^2 + |C_R'|^2 \geq 1 \Rightarrow |C_i'|^2 \leq 1 \text{ and } |C_R'|^2 \leq 1 \]

\[ |C_i'|^2 = |C_i|^2 \quad |C_R'|^2 = |C_R|^2 \]

Dividing (1) by (2)

\[ C_i'C_i'^* + C_i'C_R'^* + C_i'C_K'^* + C_K'C_R'^* = \]

\[ C_i'C_i'^* + C_i'C_R'^* + C_i'C_K'^* + C_K'C_R'^* \]

\[ \frac{C_i'C_R'^*}{C_R'C_R'^*} + \frac{C_i'C_K'^*}{C_K'C_K'^*} = \frac{C_i'C_K'^*}{C_K'C_K'^*} + \frac{C_i'C_R'^*}{C_R'C_R'^*} \]

\[ \frac{C_i}{C_R} + \left(\frac{C_i}{C_R}\right)^* = \frac{C_i}{C_K} + \frac{C_i}{C_R} \]

\[ \text{Re} \left( \frac{C_i}{C_R} \right) = \text{Re} \left( \frac{C_i}{C_K} \right) \]

\[ i \equiv \frac{C_i}{C_R} = \alpha + i \beta \quad \frac{C_i}{C_K} = \alpha + i \beta \]

\[ \text{since} \quad \left| \frac{C_i}{C_K} \right|^2 = \left| \frac{C_i}{C_R} \right|^2 \]

\[ |C_i|^2 = |\langle n_i | n_i \rangle|^2 = |\langle \gamma' | \gamma' \rangle|^2 = |\langle \gamma' | \gamma' \rangle|^2 = |C_i'|^2 \]

\[ \alpha^2 + \beta^2 = \alpha^2 + \beta^2 \]

\[ \Rightarrow \beta' = \pm \beta \]
This means that either
\[
\frac{C_i}{C_R} = \frac{C_i'}{C_R'} \quad \frac{C_i}{C_e} = \left(\frac{C_i'}{C_e'}\right)^* \]

this can be done for every \(k\).

Next we show that we must make the same choice for all \(k\).

To show this assume
\[
\frac{C_i}{C_R} = \frac{C_i'}{C_R'} \quad \frac{C_i}{C_e} = \left(\frac{C_i'}{C_e'}\right)^* \]

and let
\[
|\psi> = \frac{1}{\sqrt{3}} \left( |\Psi_i> + |\Psi_k> + |\Psi_e> \right) \]

obviously
\[
|\psi'> = a_i' |\Psi_i'> + a_k' |\Psi_k'> + a_e' |\Psi_e'> \]

from what we just showed
\[
\frac{a_i'}{a_i} = \left(\frac{a_i}{a_i'}\right)^* \quad \frac{a_k'}{a_i} = \left(\frac{a_k}{a_i'}\right)^* \quad \frac{a_e'}{a_i} = \left(\frac{a_e}{a_i'}\right)^* \]
since both ratios are real
the \( \alpha'_c \) differ from the \( \alpha_c \) by a common phase

\[
\alpha'_c = \alpha'_c \left( 1\psi'_c > + \frac{a_k}{a'_c} 1\psi'_k > + \frac{a_k}{a'_k} 1\psi'_e > \right)
\]

\[
= \alpha'_c \left( 1\psi'_c > + \frac{a_k}{a'_c} 1\psi'_k > + \frac{a_k}{a'_k} 1\psi'_e > \right)
\]

where \( |\alpha'_c| = |\alpha_c| = \frac{1}{\sqrt{3}} \) \( \alpha'_c = \frac{e^{i\phi}}{\sqrt{3}} \) for
some common phase

Consider

\[
|<\psi' | x'>|^2 = |<\psi' | x'>|^2
\]

\[
\frac{1}{3} \left| C_i + C_e + C_k \right|^2 = \frac{1}{3} \left| C'_i + C'_e + C'_k \right|^2
\]

\[
\frac{1}{3|C_i|^2} \left| 1 + \frac{C_e}{C_i} + \frac{C_k}{C_i} \right|^2 = \frac{1}{3|C'_i|^2} \left| 1 + \frac{C'_e}{C'_i} + \frac{C'_k}{C'_i} \right|^2
\]

assume by contradiction there
is a pair where \( \frac{C'_e}{C'_i} = \frac{C_e}{C_i} \)

\[ C'_k = \left( \frac{C_k}{C_i} \right) \] where both quantities
are complex.
\[
\left(1 + \frac{C_p}{C_i}\right)\left(\frac{C_R}{C_i}\right)^* + \left(1 + \frac{C_p}{C_i}\right)\left(\frac{C_k}{C_i}\right) =
\]
\[
\left(1 + \frac{C_p}{C_i}\right)\left(\frac{C_R}{C_i}\right)^* + \left(1 + \frac{C_p}{C_i}\right)^*\left(\frac{C_R}{C_i}\right)^*
\]
\[
\text{Re} \int \left(1 + \frac{C_p}{C_i}\right)\left(\frac{C_R}{C_i}\right)^* \right) = \text{Re} \int \left(1 + \frac{C_p}{C_i}\right)\left(\frac{C_R}{C_i}\right)
\]
\[
\text{Re} \left(\frac{C_R}{C_i}\right)^* = \text{Re} \left(\frac{C_R}{C_i}\right) = 0
\]
\[
\text{Re} \left(\frac{C_p C_R}{C_i^2}\right) = \text{Re} \left(\frac{C_p}{C_i}\right)
\]

Assume \( \frac{C_p}{C_i} = \alpha + i\beta \quad \frac{C_R}{C_i} = \gamma + i\delta \)

\[
\frac{C_p}{C_i} \frac{C_R}{C_i} = (\alpha + i\beta)(\gamma + i\delta) = \alpha \gamma - \beta \delta + i(\beta \gamma + \alpha \delta)
\]
\[
\frac{C_p}{C_i} \frac{C_R}{C_i} \ast = (\alpha + i\beta)(\gamma - i\delta) = \alpha \gamma + \beta \delta + i(\beta \gamma - \alpha \delta)
\]

Equality of the real part means that

\[2\beta \delta = 0 \quad \text{or} \quad \text{Im} \left(\frac{C_p}{C_i}\right) \text{Im} \left(\frac{C_R}{C_i}\right) = 0 \]

which means the one of the 2 complex ratios cannot be complex.
This shows that

\[ |\psi\rangle = c_1 (|\psi_i\rangle + \frac{c_0}{c_1} |\psi_f\rangle) \]

\[ |\psi\rangle = c_1' (|\psi_i\rangle + \frac{c_0}{c_1} |\psi_f\rangle) \quad \text{or} \]

\[ = c_1' (|\psi_i\rangle + \frac{c_0}{c_1} |\psi_f\rangle^*) \]

\(c_1'\) includes an arbitrary phase.

We can choose \(c_1' = c_1\) or \(c_1' = c_1^*\) to fix the phase.

\[ U |\psi\rangle = |\psi_i\rangle = 2 c_1 |\psi_i\rangle = 2 c_1 |\psi_i\rangle \]

\[ U |\psi\rangle = |\psi_f\rangle = 2 c_1^* |\psi_f\rangle = 2 c_1^* |\psi_f\rangle \]

These are the two possibilities for the vector \( |\psi\rangle \).

Next: Is this possible that \(U\) maps some vectors linearly and others antilinearly?
\[ U_1 \Psi_n \rangle = \sum a_n U_1 \Psi_n \rangle \\
U_1 \Psi_n \rangle = \sum b_n^* U_1 \Psi_n \rangle \]

where \( a_n \) and \( b_n \) do not all have the same phase.

We show that the only way that this can be true if the phases are all the same in one of these vectors is to consider

\[ | \langle \Psi_n | \Psi_n' \rangle |^2 = | \langle \Psi_n | \Psi_n' \rangle |^2 \]

\[ \left| \sum a_n^* b_n \right|^2 = \left| \sum a_n^* b_n \right|^2 \]

\[ \sum_{mn} a_n^* a_m b_n b_m^* = \sum_{mn} a_n^* a_m b_n b_m^* \]

Subtracting

\[ \sum_{mn} a_n^* a_m (b_n b_m^* - b_n^* b_m) = \]

\[ \sum \frac{1}{2} (a_n^* a_m - a_m^* a_n)(b_n b_m^* - b_n^* b_m) . \]
\[-\frac{1}{2} \sum_{mn} \text{Im} (a_n^* a_m^*) \text{Im} (b_n^* b_m) = 0.\]

It is true that this vanishes if all of $a_m$'s or $b_n$ have the same phase.

We need to argue that this holds for every pair as before - we assume that at least for some pair $\text{Im} (a_n^* a_m) \text{Im} (b_n^* b_m) \neq 0$.

Introduce
\[
|\Psi_{nm}\rangle = C_n |\psi_n\rangle + C_m |\psi_m\rangle
\]

where the $C_k$'s have different phases.

Compute
\[
|\langle \varphi_a | \Psi_{nm} \rangle| = |\langle \varphi_a | \Psi'_{nm} \rangle| \\
|\langle \varphi_b | \Psi_{nm} \rangle| = |\langle \varphi_b | \Psi'_{nm} \rangle|
\]

(1) Assume that $|\Psi'\rangle = U |\Psi\rangle = C_n |\psi_n\rangle + C_m |\psi_m\rangle$ (linear case). \[\rightarrow\]
\[-\frac{1}{2} \text{Im} (C_n^* C_m) \text{Im} (b_n^* b_m) + \]
\[-\frac{1}{2} \text{Im} (C_m^* C_n) \text{Im} (b_n^* b_m) = 0\]
these add (products of odd terms)

\[ \text{Im} (c_n^* c_m) \text{Im} (b_n b_m) = 0 \]

Since \( \text{Im} (c_n^* c_m) \neq 0 \) by assumption

\[ \Rightarrow \text{Im} (b_n b_m) = 0 . \]

This show all pairs have the same phase - which is a contradiction.

\[ : (\psi_a) \to (\psi_b) \text{ both behave like } |k\rangle \]

under the action of \( U \).

Similarly - assume \( |k_{nm}\rangle \) transforms like \( |\psi_b\rangle \) then

\[ \text{Im} (c_n^* c_m) \text{Im} (a_n^* a_m) = 0 . \]

the components of \( |\psi_a\rangle \) all have the same phase
again \( U \) either acts linearly on all 3 vectors or antilinearly on all 3 vectors.
what we have learned

Theorem

\[ |\psi\rangle \rightarrow |\psi'\rangle = U |\psi\rangle \]

is unitary and linear for all vectors.

\[ |\psi\rangle \rightarrow |\psi'\rangle = U |\psi\rangle \]

is antiunitary and antilinear for all vectors.

Anti-linear case

\[ |\psi_n\rangle \rightarrow |\psi'_n\rangle = U |\psi_n\rangle \]

\[ |\phi\rangle = \sum a_n |\psi_n\rangle \]

\[ |\chi\rangle = \sum b_n |\psi_n\rangle \]

\[ |\phi'\rangle = \sum a_n^* |\psi'_n\rangle \]

\[ |\chi'\rangle = \sum b_n^* |\psi'_n\rangle \]

\[ \langle \phi' | \chi' \rangle = \sum a_n b_n^* \langle \psi'_n | \psi'_n \rangle \]

\[ = \sum a_n b_n^* \langle \psi_n | \psi_n \rangle \]

\[ = \langle \chi | \phi \rangle \]
using antilinear operators

1. always use a basis
2. transformed basis is orthonormal
3. operator extends to all vectors by antilinearly

\[
U \left( \sum c_n |\Psi_n> \right) = \sum c_n^* U |\Psi_n> = \sum c_n^* |\Psi_n> 
\]
So far we have considered symmetries implemented by unitary transformations. There are some that require antiunitary transformations.

One of the operators is time reversal.

Consider what we expect:

\[ T \bar{P} T^{-1} = -\bar{P} \quad \text{(momenta change sign if the direction of time is reversed)} \]

\[ T \bar{x} T^{-1} = \bar{x} \quad \text{(coordinates remain unchanged if the direction of time is changed)} \]

\[ T(\bar{F} \times \bar{p}) T^{-1} = -\bar{F} \times \bar{p} \]

Consider:

\[ T(H) T^{-1} = T(\bar{F} \cdot \bar{P}/2m) T^{-1} \]
\[ = T(\bar{P} \cdot T^{-1}) T(\bar{P} \cdot T^{-1})/2m \]
\[ = \bar{P} \cdot \bar{P}/2m = H \]

For free particles we expect that \( H \), which is quadratic in \( \bar{P} \), to be
invariant under time reversal.

but

\[ T \, U(t) \, T^{-1} = U(-t) \]

\[ U(t) = e^{-iHt/\hbar} \]

\[ = 2 \left( \frac{-i\varepsilon}{\hbar} \right)^n H^n \]

\[ T \, U(t) \, T^{-1} = 2 \left( T \left( \frac{-i\varepsilon}{\hbar} H \right) T^{-1} \right)^n \]

if \( THT^{-1} = H \) and we choose \( T \)

to be linear then we get

\[ T \, U(t) \, T^{-1} = U(t) \]

which is inconsistent with (1)

if \( T \) is antiunitary, then

\[ T \left( \frac{-i\varepsilon}{\hbar} H \right) T^{-1} = T \left( \frac{-i\varepsilon}{\hbar} \right)^n T^{-1} \]

\[ = i\frac{\varepsilon}{\hbar} H \]

so we can have \( T \, U(t) \, T^{-1} = U(-t) \) and

\[ THT^{-1} = H \] if \( T \) is antiunitary.
There is another reason to require \( THT^{-1} = H \) rather than \(-H\).

If the second choice the eigenvalues of the time reversed \( H \) change sign. If the original \( H \) has a spectrum that extends to \( +\infty \), the time reversed operator has one that extends to \(-\infty \). The resulting system is not stable.

Spin ½ is an interesting case

\[
\vec{S} = \begin{pmatrix} \frac{1}{2} \\
\end{pmatrix}
\]

To get \( \vec{S} \) to behave like \( \vec{I} \) we want

\[
T \vec{S} T^{-1} = -\vec{S} = (-\vec{S}_x, \vec{S}_y, -\vec{S}_z)
\]

\[
= \vec{S}_y (\vec{S}_x, \vec{S}_y, \vec{S}_z)
\]

which is directly expressed in terms of a unitary transformation \( \vec{S}_y = i e^{-\frac{i}{2} \vec{S}_y \Pi} \) and a complex conjugation.
in general we can multiply $a$ by
and phase - the result is unitary

\[ T = e^{i\theta} \sigma_y K \quad K = \text{complex conjugation} \]

\[ T^2 = (e^{i\theta} \sigma_y K)(e^{i\theta} \sigma_y K) \]
\[ = e^{i\theta} e^{-i\theta} \sigma_y \sigma_y K^2 = -I \]

This is special to spin $\frac{1}{2}$ particles.

If a system has $N$ such particles then applying $T$ give a factor $(-1)^N$

Transformation properties of wave functions:

\[ T |\bar{p}\rangle = |\bar{p}\rangle \]
\[ T |\psi\rangle = \int T (|\bar{p}\rangle \langle \bar{p}|\psi\rangle) \]
\[ = \int \langle \bar{p}|\psi\rangle \langle \bar{p}|\bar{p}\rangle \]
\[ = \int |\bar{p}\rangle \langle \bar{p}|\psi\rangle \]
\[ \langle \bar{p}|\psi\rangle \rightarrow \langle \bar{p}|\psi\rangle \]

\[ T |\chi\rangle = |\chi\rangle \]
\[ T |\psi\rangle = \int T (|\chi\rangle \langle \chi|\psi\rangle) \]
\[ = \int |\chi\rangle \langle \chi|\psi\rangle \]
$$\psi(x) \rightarrow \psi^*(x)$$

Kramers degeneracy:

(i) consider an atom with $N$ spin $\frac{1}{2}$ particles

(ii) assume $THT^{-1} = H$ (the Hamiltonian in invariant with respect to time reversal)

$$THT^{-1} |E\psi\rangle = H |E\psi\rangle$$
$$= T(E|\psi\rangle)$$
$$= E T|\psi\rangle$$

If $|\psi\rangle$ is an eigenstate so is $T|\psi\rangle$

If $N$ is odd $T^2|\psi\rangle = -|\psi\rangle$.

If $T|\psi\rangle = c |\psi\rangle$

$$T^2|\psi\rangle = T(c|\psi\rangle) = c^* T|\psi\rangle$$
$$= (cc^*) |\psi\rangle$$
$$= |\psi\rangle$$

since this must preserve norm. This contradicts $T^2|\psi\rangle = -|\psi\rangle$ which means that it a system (1) is time reversal invariant and
has an odd # of spin \( \frac{1}{2} \) particles
then, all bound states must be degenerate