Lecture 14

Time dependent perturbation theory
starting in state \( n \) - ending up
in \( m \), first order

\[ c_m(t) \approx \left( -\frac{i}{\hbar} \right) \int_0^t e^{-i(E_m - E_n)t/\hbar} \langle m | \psi(t) | n \rangle \, dt \]

one electron atom in electromagnetic field

\[ \vec{A}(x,t) = a \hat{x} e^{i(\frac{\omega}{c} z - \omega t)} \]

(electromagnetic wave moving in the
\( z \) direction)

\[ \vec{E}(x,t) = -\frac{\partial \vec{A}}{\partial t} = a \omega \hat{x} e^{i(\frac{\omega}{c} z - \omega t)} \]

\[ a \omega = 1E \text{Imax} \]

\[ \vec{B}(x,t) = \nabla \times \vec{A} = i a \frac{\omega}{c} (\hat{z} \times \hat{x}) e^{i(\frac{\omega}{c} z - \omega t)} \]

\[ = i a \frac{\omega}{c} \hat{y} e^{i(\frac{\omega}{c} z - \omega t)} \]

\[ a \omega = 1B \text{Imax} \]

The normal way to include em
interactions is using minimal
substitution
\[ H = \frac{p^2}{2m} - \frac{Ze^2}{r} \rightarrow \]
\[ = \left( \frac{\not{p} - \frac{e}{c} \not{A}}{2m} \right)^2 + eA - \frac{Ze^2}{r} \]
\[ = \frac{p^2}{2m} - \frac{Ze^2}{r} - \frac{e}{2mc} \left( A \cdot \not{p} + \not{p} \cdot A \right) + \frac{e^2}{2mc^2} \not{A}^2 + e\phi \]

In this example, \( \phi = 0 \); we assume the field is weak \( \Rightarrow \) ignore the \( \lambda^2 \) term.

\[ H = H_0 + V \]

\[ H_0 = \frac{p^2}{2m} - \frac{Ze^2}{r} \]

\[ V = -\frac{e}{2mc} \left( A \cdot \not{p} + \not{p} \cdot A \right) \]
\[ = -\frac{e}{2mc} \left( \frac{i(\omega z - \omega t)}{2} \right) P_x \]

We insert this in the expression for \( Cm(t) \):

\[ Cm(t) = (-\frac{i}{\hbar}) \int_0^t \left( e^{\frac{1}{\hbar} (Em-En)t} \right) \left( \frac{-e\hbar}{mc} \right) e^{\frac{-i\omega t}{\hbar}} \frac{i\hbar}{2} \begin{pmatrix} 1 & i\omega z \\ -i\omega z & 1 \end{pmatrix} \begin{pmatrix} m_1 e \\ P_x \\ ln \end{pmatrix} dt \]
\[ = (-\frac{i}{\hbar}) \left( \frac{e\hbar}{mc} \right) \left( e^{\frac{iEm-En}{\hbar} t} \right) \begin{pmatrix} i\frac{Em-En}{\hbar} t & i\hbar z \\ -i\hbar z & 1 \end{pmatrix} \begin{pmatrix} m_1 e \\ P_x \\ ln \end{pmatrix} \]
\[ = \frac{e\hbar}{mc} (2i) e^{\frac{iEm-En}{\hbar} t} \frac{\sin \left( \frac{Em-En}{2\hbar} t \right)}{\left( Em-En-\omega t \right)} \begin{pmatrix} m_1 e \\ P_x \\ ln \end{pmatrix} \]
This gives the transition probability

$$|\langle m|e\rangle|^2 = \frac{e^2 \alpha^2}{m^2 c^2} \frac{\sin^2 \left( \frac{E_n - E_p - \hbar \omega}{2m} t \right)}{(E_n - E_p - \hbar \omega)^2} \langle k|m|l, e\rangle \langle P_x l n \rangle$$

The calculation requires calculating the matrix element

$$i \frac{\hbar}{c} \langle m|e\rangle \langle P_x l n \rangle$$

Example 2: Magnetic resonance

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$H_0 \langle 1 \rangle = E_1 \langle 1 \rangle$$

$$H_0 \langle 0 \rangle = E_2 \langle 0 \rangle$$

$$V = -\vec{\mu} \cdot \vec{B} = B_0 \vec{\gamma} \vec{e}^{-i\omega t}$$

$$= -\frac{\gamma_0}{2m} \vec{B} - \frac{e\hbar}{4m} B_0 \vec{\gamma} \vec{e}^{-i\omega t}$$

$$= -\frac{e\hbar}{4m} B_0 e^{-i\omega t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Equations

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} - \frac{i}{\hbar} \int_0^t \begin{pmatrix} 0 & 0 \\ 0 & e^{iE_1 t / \hbar} \end{pmatrix} \left( -\frac{e\hbar B}{4m} e^{-i\omega t} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & e^{-iE_2 t / \hbar} \\ e^{-iE_2 t / \hbar} & 0 \end{pmatrix} dt' \begin{pmatrix} c_1(t') \\ c_2(t') \end{pmatrix}$$
Here we have a pair of coupled integral equations.

Assuming \( C_1(0) = 1 \) and \( C_2(0) = 0 \), the first order solution is:

\[
C_2(t) = \frac{ieB}{2\pi} \left( \frac{E_2 - E_1 - \xi \omega}{m} \right) \int C_1(t') \, dt'
\]

\[
= \frac{2\pi i eB\hbar}{4m} e^{\frac{ieB E_1 - \xi \omega t}{2\hbar}} \sin\left( \frac{E_2 - E_1 - \xi \omega t}{2\hbar} \right)
\]

\[
|C_2(t)|^2 = \frac{e^2 B^2 \hbar^2}{4m^2} \sin^2\left( \frac{E_2 - E_1 - \xi \omega t}{2\hbar} \right)
\]

Quantum mechanics of identical particles:

Two electrons can't be distinguished.

Physical state -
- One electron in state \( |\phi_{\frac{1}{2}}\rangle \)
- One electron in state \( |\phi_{-\frac{1}{2}}\rangle \)

This state can be described in the following two ways:
\[ |\Psi_1\rangle = |\vec{\psi}_{\frac{1}{2}}\rangle_1 |\vec{\psi}_{-\frac{1}{2}}\rangle_2 \]
\[ |\Psi_2\rangle = |\vec{\psi}_{-\frac{1}{2}}\rangle_1 |\vec{\psi}_{\frac{1}{2}}\rangle_2 \]

This exhibits two orthogonal vectors representing the same physical state.

We expect
\[ (|\Psi_1\rangle |\Psi_1\rangle)^2 = K |\Psi_2\rangle |\Psi_2\rangle = K |\Psi_1\rangle |\Psi_1\rangle = K |\Psi_2\rangle |\Psi_2\rangle \]

The first two probabilities are 1, the last two are 0. This is a contradiction.

Consistency requires

\[ P_{12} \text{ (exchange the electrons)} \]
\[ P_{12} |\Psi_1\rangle \propto |\Psi_2\rangle \text{ (they should be in the same ray)} \]
\[ P_{12} = 1 \]
\[ P_{12}^2 |\Psi_1\rangle = |\Psi_1\rangle \]
\[ (P_{12}^2 - 1) |\Psi_1\rangle = 0 \]

eigenvalues ± 1
consistency requires

\[ P_{12} \mid \psi_1 \rangle = \pm \mid \psi_1 \rangle \]

we can have both – otherwise

\[ \langle \psi_+ | P_{12} | \psi_- \rangle = -\langle \psi_- | \psi_+ \rangle = \langle \psi_+ | \psi_- \rangle \]

\[ 2 \langle \psi_+ | \psi_- \rangle = 0 \]

and we are back with the same problem – so we have to choose 1 or the other.

Quantum field theory gives the physically realized answer

In order for experiments in small laboratories that can be influenced by a light signal to be independent.

(1) Wave functions for systems of half integral spin identical particles should be antisymmetric with respect to exchange.

(2) Wave functions for systems of integer spin identical particles should be symmetric with respect to exchange.
example 3 particles

\( |abc> \)
\( \pm |bac> \)
\( |bca> \)
\( \pm |cba> \)
\( |cab> \)
\( \pm |acb> \)

The summetrized (antisymmetrized) state is

\[ (|abc> + |bca> + |cba> + |acb> \pm |bac> \pm |cba> \pm |bca> ) \]

Since these states are all orthogonal, the correctly symmetrized (antisymmetrized) state is

\[ |4> = \frac{1}{\sqrt{6}} (|abc> + |bca> + |cba> \pm |acb> \pm |bac> \pm |cba> ) \]

Note that if the Hamiltonian has identical particles, \( |4> \) preserves the symmetry.
we can write this in terms of permutation operators

\[
\begin{align*}
P(abc) & \quad P(\text{bac}) \\
P(abc) & \quad P(\text{acb}) \\
P(abc) & \quad P(\text{cba}) \\
\end{align*}
\]

\[
|\Psi_\text{f}\rangle = \sum \frac{1}{\sqrt{6}} (-1)^i P_i |\Psi_\text{f}\rangle
\]

to understand the sign

\[
\begin{align*}
P(abc) &= 1 \\
P(\text{bac}) &= P_{12} \\
P(\text{acb}) &= P_{13} \\
P(\text{cba}) &= P_{23} \\
P(\text{acb}) &= P_{12} \cdot P_{23} \\
P(\text{hab}) &= P_{12} \cdot P_{23} \\
\end{align*}
\]

we see the permutation matrix have an even # of exchanges have a +; the permutation with odd # of permutations have (-1) sign

\[
\sigma = \text{permutation}
\]

\[
|\sigma| = \begin{cases} 
1 & \sigma = \text{prod of even } \neq P_{11} \\
-1 & \sigma = \text{prod of odd } \neq P_{11} 
\end{cases}
\]
We show that $|G|$ is well defined

Assume by contradiction that

$$P(b) = \prod P_{ij} = \prod P_{kj} \quad \text{even odd}$$

Since $P_{ij} = 1$ multiplying in the opposite order gives

$$I = \prod (P_{ij}) \prod (P_{kj}) \quad \text{even odd}$$

Let $\psi_1$ to be antisymmetric under interchange of pair

$$\psi_1 = \prod \psi_1 = \prod (P_{ij}) \prod (P_{kj}) \psi_1 \quad \text{even odd}$$

$$= (-1) \psi_1$$

or

$$2 \psi_1 = 0$$

This contradicts $\psi_1 = 0$. This shows $|G|$ is well defined

but:

(i) The total # of transpositions that make a permutation is not unique

$$I = P_{12} P_{13} P_{23} P_{12}$$
(2) \( P_{ij} \) do not commute

\[
P_{12}P_{23} = 123 - 312 = 312 \\
P_{23}P_{12} = 213 - 132 = 132
\]

\[\therefore P_{12}P_{23} \neq P_{23}P_{12}\]

(3) There is more than one way to express a permutation

\[
123 \rightarrow P_{23}P_{12} \\
231 \rightarrow P_{13}P_{23}
\]

for \( n \) identical particles 1, 2, \ldots, \( n \)

\( n \) places to move 1,
\( n-1 \) places to move 2

\[ \ldots \]

\( 1 \) place to move \( n \)

\[ n! \]

distance permutations of \( n \) objects

\[
S = \frac{1}{n!} \sum_{\sigma \in S(n)} P_{\sigma}
\]

\[
A = \frac{1}{n!} \sum_{\sigma \in A(n)} (-1)^{\ell+1} P_{\sigma}
\]

\[14.10\]
\[ S^2 = \sum \frac{1}{(n!)^2} \sum \frac{1}{c} \ P_0 \ P_0' \]
\[ = \sum \frac{1}{(n!)^2} \sum \frac{1}{c} \ P_0 \ P_0' \]

for fixed \( c' \) \( \sum P_0 \ P_0' = \sum P_0 = -ic \) it goes over all permutations

\[ = \sum \frac{1}{(n!)^2} \sum \frac{1}{c} \sum P_0'' = \sum \frac{n!}{c} \ P_0'' = S \]

Similarly

\[ A^2 = \sum \frac{1}{(n!)^2} (-)^{l+1} \ P_0 \ P_0' \]

\[ = \sum \frac{1}{(n!)^2} (-)^{l} \ P_0'' = A \]

and

\[ A \cdot S = \sum \frac{1}{(n!)^2} (-)^{l} \ P_0 \ P_0' \]

Note \( n! \) is even unless \( n=1 \) - half of the permutations are even
half are odd

Let \( G_e \) even \( \rightarrow G_e = G_{e,0} \)

\( G_e \pm G_e' = \) \( G_e \pm G_e' \) otherwise

\( G_e = G_e' = 0 \) \( \rightarrow G_e = G_e' \) \( \pm 1 \) \( G_e = G_e' \)
this shows that $A$ and $S$ are projection operators

For Hamiltonian with identical particles $P_{ij} H = H P_{ij} = I$

$A H = HA$, $S H = H S$

since $H$ commutes with both of these, there are simultaneous eigenstates $H$ and $A$ or $H$ and $S$

The physically relevant states are the states with the desired symmetry.

*occupation representation*

Let $|\Phi_n\rangle$ be a complete set of orthonormal states of a single particle

For identical particles all that can be measured are

$\#$ particles in state $|\Phi_1\rangle$

$\#$ particle, in state $|\Phi_2\rangle$

;
define

\[ |m_1, m_2, \ldots, m_k, \ldots \rangle \]

\[ m_k = \text{# of particles in state } k \]

for a system of \( N \) particles

\[ N = \sum_{k=1}^{\infty} m_k \]

so all but a finite number of the \( m_k \) will be 0.

\[ |\Phi_{m_1}, \Phi_{m_2}, \ldots, \Phi_{m_k} \rangle \]

If the particles are fermions then \( m_k \) can only be 0 or 1.

\[ |1 1 0 1 0 0 \rangle = \left( \Phi_1(x_1) \Phi_3(x_2) + \Phi_3(x_1) \Phi_1(x_3) \right) \frac{1}{\sqrt{2}} \]

we need a convention for the signs.
Let
\[ |0\rangle = \underbrace{1 0 0 0 \cdots}_n \]
denote the 0 particle state. Next we define operators - for integer spin particles (called bosons).

$\hat{a}_m^+$ creates a particle in state $\hat{\phi}_n$\
\[ \hat{a}_m^+ |0\rangle = \underbrace{1 0 \cdots 1 0}_{\text{min}} |0\rangle \]
\[ \langle 0 | \hat{a}_m^+ \hat{a}_m^+ |0\rangle = \langle 0 | \underbrace{1 \cdots 1 0}_{\text{min}} |1\rangle = 1 \]

This suggests that $\hat{a}_m^+$ removes a particle in state $\hat{\phi}_m$\
\[ [\hat{a}_m^+, \hat{a}_n] = \delta_{mn} \]
\[ \left( \hat{a}_m^+ \right)^n |0\rangle = \underbrace{1 0 \cdots 0}_{\text{min}} |n\rangle \]

This gives a state with $n$ particles in state $\hat{\phi}_m$ (1 symmetric) and normalized to unity.

\[ \hat{a}_m^+ \hat{a}_n^+ |0\rangle = \hat{a}_n^+ \hat{a}_m^+ |0\rangle = \frac{1}{\sqrt{2}} \left( \hat{\phi}_m(x_1) \hat{\phi}_n(x_2) + \hat{\phi}_n(x_1) \hat{\phi}_m(x_2) \right) \]
For half integral spin particles we introduce $b_n^+ b_n^+ b_m^+ b_m^+ = 0$.

This gives $b_n = 0$ (can't put 2 particles in state n) $b_n^+ b_m^+ = - b_m^+ b_n^+$ which ensures that the wave function changes sign.

$\{A, B^\dagger\} = AB + BA$

$\{b_m^+, b_n^+\} = S_{mn}$

$\{b_m^-, b_n^\dagger\} = 0$

$\{b_m^+, b_n^\dagger\} = 0$

In order to relate these operators to the occupation number representation order states so lowest $m$ comes below higher $m$.

$\begin{align*}
  b_3^+ b_1^+ |0> &= + \frac{1}{\sqrt{2}} \left( \Phi_1(1) \Phi_3(2) - \Phi_3(1) \Phi_1(2) \right) \\
  b_1^+ b_3^+ |0> &= - b_3^+ b_1^+ |0>
\end{align*}$

Once we have a convention everything else is automatic. The operators take care of the signs.