Lecture 24

Coulomb Scattering

The Coulomb potential has a long range - sufficiently long that particles are not entirely free no matter how far they are from each other.

We treat this problem by solving the Schrödinger equation for positive energy eigenvalues. We try to identify solutions with modified scattering asymptotic condition:

\[ H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{Z_1 Z_2 e^4}{r} \]

It turns out that even though this problem has spherical symmetry it is useful to solve it using a parabolic coordinate system:

\[ m = \frac{1}{2} (r - z) \quad \quad \quad \quad x = 2 \sqrt{m_1} \cos \phi \]
\[ n = \frac{1}{2} (r + z) \quad \quad \quad \quad y = 2 \sqrt{m_1} \sin \phi \]
\[ t + n \phi = \frac{\nu}{x} \quad \quad \quad \quad z = s - \eta \]
\[ r = \frac{\nu}{x} + \eta. \]

We write the Laplacian in parabolic coordinates.
After some work one can show (H \omega)

\[ \nabla^2 = \frac{1}{(3+n)} \left\{ 3 \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} + \eta \frac{\partial^2}{\partial \eta^2} + \frac{2}{\partial \eta} + \frac{1}{45 n} \frac{\partial^2}{\partial \varphi^2} \right\} \]

\[ \frac{1}{\hbar^2} = \frac{1}{(3+n)} \]

Thus the Schrödinger equation becomes

\[ -\frac{\hbar^2}{2m} \frac{1}{(3+n)} \left\{ 3 \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} + \eta \frac{\partial^2}{\partial \eta^2} + \frac{2}{\partial \eta} + \frac{1}{4} \left( \frac{1}{3} \right) \frac{\partial^2}{\partial \varphi^2} \right\} \psi = \frac{k^2}{2m} \psi \]

where \( k^2 / 2m \) is the energy eigenvalue.

Multiply both sides by \( \hbar^2 / k^2 (3+n) \)

\[ \left( 3 \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} + \eta \frac{\partial^2}{\partial \eta^2} + \frac{2}{\partial \eta} + \frac{1}{4} \left( \frac{1}{3} + \frac{1}{n} \right) \frac{\partial^2}{\partial \varphi^2} \right) \psi \]

\[ -\frac{2n \frac{\partial^2}{\partial \xi^2} \psi}{k^2} \]

\[ + \frac{k^2}{\hbar^2 (3+n)} \psi = 0 \]

Define \( k = k / \hbar \); \( \lambda = \frac{2e Z \frac{\partial^2}{\partial \xi^2}}{k^2} \)

Since our solution should be periodic in \( \varphi \) we assume

\[ \psi = F(\eta, \xi) e^{i \lambda \varphi} \]

Then \( \frac{\partial^2}{\partial \varphi^2} \rightarrow -m^2 \)
If the incident beam is in the $z$ direction then we expect that
the solution will be independent of $\phi = 0 \quad m = 0$

In the standard theory - for an incoming wave in the
$z$ direction we expect

$$
\psi \rightarrow e^{\frac{ikz}{\hbar}} + \frac{e^{i kr/\hbar}}{r} F(r, z)
$$

$$
= e^{\frac{ik (z - m)}{\hbar}} + \frac{e^{i k (z + m) / \hbar}}{z + m} F(z, m)
$$

$$
= e^{\frac{i k z / \hbar}{}} \left( e^{-i k n / \hbar} + \frac{e^{i k n / \hbar}}{z + n} F(z, n) \right)
$$

As we mentioned initially -
this boundary condition has
to be modified. Motivated by
this we try a solution of
the form

$$
\psi (n, z, \phi) = e^{\frac{i k z}{\hbar}} g(n)
$$

that has energy eigenvalue $k^2 / 2m > 0$
It turns out to be more useful to write
\[ \psi(m \geq 0) = e^{-\frac{ik \xi}{\hbar}} \psi_F(n) \]
\[ \left( \frac{\partial^2}{\partial \xi^2} + \frac{2}{\xi} \right) e^{ik \xi/\hbar} = \left( -\frac{k^2}{\hbar^2} \frac{\xi}{\hbar^2} + \frac{k}{\hbar} \right) e^{ik \xi/\hbar} \]

The differential equation becomes
\[ \left( -\frac{k^2}{\hbar^2} \frac{\xi}{\hbar^2} + \frac{k}{\hbar} \right) e^{ik \xi/\hbar} \psi_F(n) \]
\[ - \frac{2 \mu Z \xi e^{i \xi/\hbar}}{\hbar^2} e^{i \xi/\hbar} \psi_F(n) + \frac{k^2}{\hbar^2} \frac{\xi}{\hbar^2} e^{i \xi/\hbar} \psi_F(n) = 0 \]

Note
\[ \left( \eta \frac{d^2}{dn^2} + \frac{d}{dn} \right) e^{i \kappa n/\hbar} F = \]
\[ \eta \left( -\frac{k^2}{\hbar^2} F' + F'' \right) - \frac{i k}{\hbar} F + F' \]
\[ \eta F''(-2i \frac{\kappa}{\hbar} n + i) F'+ \left[ -\frac{k^2}{\hbar^2} \eta - \frac{i k}{\hbar} \right] F = \]

Let \[ \lambda = \frac{2 \mu Z \xi e^{i \xi/\hbar}}{\hbar^2} \quad \kappa = \frac{k}{\hbar} = \eta \]

\[ \eta F'' + (1 - 2i \kappa n) F' - \kappa^2 n F - i \kappa F - \lambda F + \kappa^2 n F + i \kappa F = 0 \]
\[ \eta F'' + (1 - 2i \kappa n) F' + \lambda F = 0 \]
Note that
\[(x F'' + (\nu+1-x) F') + \nu F = 0\]
has solution,
\[L^\nu_n(x) = \frac{1}{n!} x^{-\nu} e^x \frac{d}{dx} \left( x^{\nu+n} e^{-x} \right)\]
To make the connection let
\[x = 2i\kappa n\]
\[\frac{d}{dn} = \left(2i\kappa\right) \frac{d}{dx}\]
\[\nu = \frac{1}{2i\kappa}\]
\[x \frac{1}{(2i\kappa)^2} \left(2i\kappa\right)^2 \frac{d^3}{dx^2} F + (1-x) \frac{d^2}{dx^2} F + \frac{\lambda}{2i\kappa} F = 0\]
\[x \frac{d^2}{dx^2} F + (1-x) \frac{d}{dx} F - \frac{\lambda}{2i\kappa} F = 0\]
this suggests
\[x = 2i\kappa n\]
\[\nu = \frac{i}{4\kappa}\]
\[\lambda = \nu\]
\[F = L^{\nu}_{i\kappa} (2i\kappa n)\]
\[\psi = e^{-e} L^{\nu}_{i\kappa} (2i\kappa n)\]
The polynomials are one solution of Laguerre's equation - there are 2 independent solutions. We need to choose one with reasonable asymptotic conditions.

Recall

\[ L_n^\alpha(x) = \frac{1}{n!} e^x \frac{d}{dx} \left( x^n e^{-x} \right) \]

\[ = e^x \oint \frac{t^n e^{-t}}{(t-x)^{n+1}} dt \]

where the contour goes around the point \( x \). We can use this to extend this expression to non integer values of \( n \).

When \( n \) complex \( t^n \) and \((t-x)^{(-n-1)}\) are multi-sheeted function.

We can choose a cut that connects the points where they are 0. Then a suitable contour...
This gives a solution - we need to check BC

\[ L_{i\lambda/2k} (2i\kappa n) = e^{2i\kappa n} \frac{i}{2\pi i} \int \frac{e^{-t} t^{i\lambda/2k}}{(t - 2i\kappa n)^{1+i\lambda/2k}} dt \]

This can be evaluated by deforming the contour

![Diagram](image)

For the path A

\[ t = s + 2i\kappa n \]

For the path B

\[ t = s \]

For the path C

\[ t = u + i s \quad (e^{-t} \text{ kills this at } u \to \infty) \]

The integrals become

\[ A = \frac{e^{2i\kappa n}}{2\pi i} \left\{ \int_{\infty}^{0} \frac{e^{-s} (s + 2i\kappa n)^{i\lambda/2k}}{(s)^{1+i\lambda/2k}} ds \right. \]

\[ + \left. \int_{0}^{\infty} \frac{e^{-s} (e^{2\pi i s} + 2i\kappa n)}{(e^{2\pi i s})^{1+\lambda/2k}} \right\} \]
\[ B = \frac{e^{2i \lambda \eta}}{2\pi i} \left\{ \int_0^\infty \frac{e^{-s}}{(s - 2i \lambda \eta)^{1+i/\lambda \eta}} \, ds + \int_0^\infty \frac{e^{-s} (e^{2\pi i s})^{i/\lambda \eta}}{(e^{2\pi i s} - 2i \lambda \eta)^{1+i/\lambda \eta}} \, ds \right\} \]

we approximate for large \( \eta \) because of the factor \( e^{-s} \)

\[ (s + 2i \lambda \eta) \rightarrow (2i \lambda \eta) \text{ etc.} \]

\[ A \rightarrow \frac{e^{2i \lambda \eta}}{(2\pi i)} \left\{ e^{-2i \lambda \eta} \int_0^\infty \frac{e^{-s}}{s^{1+i/\lambda \eta}} \, ds + \left( 2i \lambda \eta \right)^{i/\lambda \eta} \int_0^\infty \frac{e^{-s}}{(e^{2\pi i s})^{1+i/\lambda \eta}} \, ds \right\} \]

\[ B \rightarrow \frac{e^{2i \lambda \eta}}{2\pi i} \left( \frac{1}{(-2i \lambda \eta)^{1+i/\lambda \eta}} \int_0^\infty e^{-s} s^{i/\lambda \eta} \right. \]
\[ + \left. \int_0^\infty e^{-s} (e^{2\pi i s})^{i/\lambda \eta} \right\} \]

\textit{note that} \( (e^{2\pi i})^{i/\lambda \eta} = e^{-\pi/\lambda \eta} \)

\[ (2i \lambda \eta)^{i/\lambda \eta} = (e^{i \eta/(2i \lambda \eta)})^{i/\lambda \eta} \]
\[ = e^{i/\lambda \eta \ln(2i \lambda \eta)} \]