Inverse scattering

To what extent is the potential determined by a complete measurement of the S matrix elements?

Let $A$ be a unitary operator satisfying

1. $AA^* = A^*A = I$
2. $\lim_{t \to \pm \infty} \| (I - A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$

for both time limits

Define

$\Delta = A - I$

$H' = AHA^* = (I + \Delta)(H_0 + V)(I + \Delta^*)$

$= H_0 + A V A^* + \Delta H_0 + H_0 \Delta^* + \Delta H \Delta^*$

$V' = A V A^* + \Delta H_0 + H_0 \Delta^* + \Delta H \Delta^*$

$H = H_0 + V$

$H' = H_0 + V'$

Consider

$$\lim_{t \to \pm \infty} e^{-iH_0 t/\hbar} e^{-iH_0 t/\hbar} |\psi\rangle =$$

$$\lim_{t \to \pm \infty} e^{-iAHA^* (I + \Delta) (H_0 + V) (I + \Delta^*)} |\psi\rangle =$$
\[ \lim_{t \to \pm \infty} e^{\pm iHt/n} = \langle 4 \rangle \]
\[ \lim_{t \to \pm \infty} A e^{\pm iHt/n} A^\dagger (1-A)e^{\mp iHt/n} = \langle 4 \rangle \]

The limit of the first term exists and is
\[ A \Omega_+ \langle 4 \rangle \]
The limit of the second term is bounded by
\[ \lim_{t \to \pm \infty} \| A e^{\pm iHt/n} A^\dagger (1-A)e^{\mp iHt/n} \| = \lim_{t \to \pm \infty} \| (1-A)e^{\mp iHt/n} \| = 0 \]
by assumption.

It follows that
\[ (i) \lim_{t \to \pm \infty} e^{\pm iHt/n} = \Omega_+ \text{ exists and} \]
\[ (ii) \lim_{t \to \pm \infty} e^{\pm iHt/n} = A \Omega_+ \]

\[ \Omega_\pm = \Omega_\pm \]

and
\[ \Omega_\pm = \Omega_+ \Omega_- = \Omega_+ A^\dagger A \Omega_- = \Omega_+ \Omega_- \in S \]

we see that there are at least as many equivalent potentials as there are unitary operators satisfying.
\[
\lim_{t \to \pm \infty} \langle (A - I) e^{-iH_0 t / \hbar} \rangle = 0
\]

The converse is also true:

\[ H = H_0 + V' \quad H = H_0 + V \]

(1) Have the same \# of bound states with the same eigenvalues and

(2) \[ S' = \Omega_+ \Omega_- = \Omega_+ \Omega_- = S \]

Given these conditions we can construct an operator \( A \) satisfying the asymptotic condition:

\[ \Omega_+ \Omega_- = \Omega_+ \Omega_- \]

\[ \Omega_+ \Omega_+ = \Omega_+ \Omega_+ \]

define

\[ A = \Omega_+ \Omega_- + \sum | \beta_n \rangle \langle \beta_n | \]

\[ = \Omega_+ \Omega_- + \sum | \beta_n \rangle \langle \beta_n | \]
0) Unitarity

\[ AA^\dagger = \left( \Omega^+_1 \Omega^+_1 + \sum \left( \Omega^+_m + \sum \Omega^+_n \right) \right) \]

\[ \times \left( \Omega^+_1 \Omega^+_1 + \sum \left( \Omega^+_m + \sum \Omega^+_n \right) \right) \]

\[ = \left( \Omega^+_1 \Omega^+_1, \Omega^+_1 \Omega^+_1 + \sum \left( \Omega^+_n \right) \right) \]

\[ \times \sum \left( \Omega^+_n \right) \]

\[ \Omega^+_1 \Omega^+_1 = \lim_{\eta \to 0} iA^\dagger \eta e \Omega^+_1 \Omega^+_1 \eta A \]

\[ \lim_{\eta \to 0} iA^\dagger \eta e \Omega^+_1 \Omega^+_1 \eta A \]

\[ = \lim_{\eta \to 0} iA^\dagger \eta e \Omega^+_1 \Omega^+_1 \eta A \]

\[ \text{set} \quad t = \eta = 0 \]

\[ \Omega^+_1 \Omega^+_1 \text{ projection on subspace of bound states} \]

\[ AA^\dagger = \left( 1 - \sum \left( \Omega^+_n \right) \right) + \sum \left( \Omega^+_n \right) = I \]

0) Transformation property

\[ AH = \left( \Omega^+_1 \Omega^+_1 + \sum \left( \Omega^+_n \right) \right) H \]

\[ = \left( \sum \left( \Omega^+_1 \Omega^+_n + \sum \left( \Omega^+_n \right) \right) \right) \]

\[ = H^{\dagger} ( \Omega^+_1 \Omega^+_1 + \sum \left( \Omega^+_n \right) ) \]

\[ = H^{\dagger} A \]

\[ \therefore \quad AHA^\dagger = H^{\dagger} \]
Asymptotic condition

\[
\lim_{t \to \pm \infty} \| (A - I) e^{-iHt/n} 1 \|^2 = \\
\lim_{t \to \pm \alpha} \| (\Omega_2^{\dagger} - I + \sum |B_n\rangle \langle B_n|) e^{-iHt/n} 1 \|^2 \\
\lim_{t \to \pm t_0} \| (\Omega_2^{\dagger} - I) e^{-iHt/n} 1 \|^2 + \sum |KB_n| e^{-iHt/n} 1 \|^2 \\
\text{the terms } |B_n| e^{-iHt/n} 1 \text{ vanish as } t \to \infty \text{ by the Riemann-Lebesgue lemma.}
\]

\[ f(E) = \int \Theta(E) \langle B_n | E, 1 \rangle \langle dE | 1 \rangle \text{ d}E \]

\[ \int_{-\infty}^{\infty} f(E) \text{ d}E = \langle B_n | 1 \rangle \text{ which is finite} \]

\[ \int_{-\infty}^{\infty} f(E) e^{-iEt/n} \text{ d}E \to 0 \text{ as } t \to \infty \]

What remains is

\[ \lim_{t \to \pm \infty} \| (\Omega_2^{\dagger} - I) e^{-iHt/n} 1 \|^2 = \\
\lim_{t \to \pm t_0} \| e^{-iHt/n} (\Omega_2^{\dagger} - I) e^{-iHt/n} 1 \|^2 = 0 \\
\| (\Omega_2^{\dagger} \Omega_2^{\dagger} - \Omega_2^{\dagger} - \Omega_2^{\dagger}) 1 \|^2 = 0 \\
\]
This shows that every potential that gives the same S matrix must be related this way.

It is important that \( \| (A-I)e^{-iH \tau_3} 1 \| \to 0 \)

for both limits.

Note unitarity is not enough—consider 2 repulsive rotationally invariant potentials

\[
H = \sum \frac{k}{2\alpha} <\text{e}^m(k) | k^2 \text{dk} \frac{k}{2\alpha} <\text{e}^m(k) |
\]

\[
H' = \sum \frac{k'}{2\alpha} <\text{e}^m(k') | k' \text{dk'} \frac{k'}{2\alpha} <\text{e}^m(k') |
\]

\[
A_+ = \sum <\text{e}^m(k^+) | k^2 \text{dk} <\text{e}^m(k^+) |
\]

is unitary and satisfies

\[
\Lambda H A_+^\dagger = H'
\]

but

\[
A_+ \neq A_- \quad s \neq s'
\]

Scattering with spin. For particles with spin

\[
\langle p, u, p, u, l, v | p', u', p', u' \rangle = s(\vec{p} + \vec{p}_2 - \vec{p'} + \vec{p'}_2) \langle \vec{k} u, u, l, v | \vec{k'}, u', u' \rangle
\]
The Lippmann–Schwinger equation is an abstract equation which works for any basis

\[
\langle \bar{\kappa}u_1 u_2 | \Gamma \left( \frac{\kappa^2}{2u} + i\epsilon \right) | \bar{\kappa}\mu_1 u_1 \rangle = \\
\langle \bar{\kappa}u_1 u_2 | \Gamma | \bar{\kappa}\mu_1 u_1 \rangle + \\
\sum_{\mu''\mu'''} \int d^3 k'' \frac{\langle \bar{\kappa}u_1 u_2 | \Gamma | \bar{\kappa}\mu'' u'' \rangle \langle \bar{\kappa}\mu'' u'' | \Gamma \left( \frac{\kappa^2}{2u} + i\epsilon \right) | \bar{\kappa}\mu_1 u_1 \rangle}{(\kappa''^2 - \kappa''^2/2u + i\epsilon)}
\]

If the potential is rotationally invariant then

\[
\langle \bar{\kappa}u_1 u_2 \rangle = \langle \bar{\kappa}u_1 u_2 | \langle \mu'' | \mu'' \rangle \langle \mu_1 | \mu_1 \rangle \rangle
\]

\[
= \langle \bar{\kappa}u_1 u_2 | \langle \mu'' | \mu'' \rangle \langle \mu_1 | \mu_1 \rangle \rangle \times
\]

\[
Y_m(\bar{k}) \langle \mu'' | \mu'' \rangle \langle \mu_1 | \mu_1 \rangle \rangle
\]

The relevant equations are then

\[
\langle \kappa (s) | t_y (\bar{k}) | \kappa' (s') \rangle \\
\langle \kappa (s) | t_y (\bar{k}) | \kappa' (s') \rangle = \\
\langle \kappa (s) | \Gamma | \kappa' (s') \rangle + \\
\int d^3 k'' \sum_{s''s'''} \frac{\langle \kappa (s) | \Gamma | \kappa' (s') \rangle \langle \Gamma | \mu'' \rangle \langle \mu'' | \mu'' \rangle \rangle \times
\]

\[
\frac{\langle \kappa''(s'') | t_y (\bar{k}) | \kappa' (s') \rangle}{(\kappa''^2 - \kappa''^2/2u + i\epsilon)}
\]
where
\[ \sum_{s} \left( \sum_{s'=1}^{15} \sum_{s''=1}^{15} \sum_{s'''}=14+1} \right) \]
\[ \frac{\Delta}{Z} - \frac{\Delta}{Z} \]

once this is solved the full transition matrix element becomes
\[ \langle \vec{k}', \vec{u}', \vec{v}' | T \left( \frac{\vec{p}}{2i} + i\epsilon \right) | \vec{k}, \vec{u}, \vec{v} \rangle = \]
\[ \sum Y_{s}^{m}(\hat{k}) \langle s, u, s', u', 1 | s', u, s'' | s' = 1, u, s'' \rangle \]
\[ \langle \vec{k}' \rangle \langle \vec{k}' \rangle \langle \hat{\vec{k}}' | T \left( \frac{\vec{p}}{2i} + i\epsilon \right) | \vec{k}'' \rangle \]
\[ \langle s, u | s', u' | s', u' \rangle \langle s, u | s', u' | s', u' \rangle \]
\[ \sum' \sum'' \sum''' \left( \hat{k} \right) \]

these are the kinds of integral equations that have to be solved

the scattering amplitude becomes
spin dependent
\[ F(\vec{k}', \vec{u}', \vec{v}' | \vec{k}, \vec{u}, \vec{v} \rangle = \]
\[ -(2\pi)^{2} m k \langle \vec{k}', \vec{u}', \vec{v}' | T \left( \frac{\vec{p}}{2i} + i\epsilon \right) | \vec{k}, \vec{u}, \vec{v} \rangle \]

when spin is involved this quantity is called
\[ M(\vec{k}', \vec{u}', \vec{v}' | \vec{k}, \vec{u}, \vec{v} \rangle = F(\vec{k}', \vec{u}', \vec{v}' | \vec{k}, \vec{u}, \vec{v} \rangle \]
\[ \frac{d\sigma}{d\Omega} = |M|^2 \]

This assumes that the beam, target and each final particle has a specific \( \mathbb{Z} \) component of angular momentum.

This would be a difficult experiment.

In elastic scattering we assume the beam and target particles have random spins.

The probability in any one of them is \( \frac{1}{2s+1} \) \( \frac{1}{2s'+1} \)

Experimentally - all outcomes are detected

\[ \left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{1}{2s+1} \frac{1}{2s'+1} \sum M(\beta, \epsilon, \epsilon', \omega, \omega') M^*(\omega, \omega', \epsilon, \epsilon') \]

If we think of \( M \) as a matrix

This has the form

\[ \left( \frac{d\sigma}{d\Omega} \right)_{el} = \frac{1}{2s+1} \frac{1}{2s'+1} \text{Tr}(M M^*) \]
mum generally the beam and target might be partially
polarized - this can be defined by a density matrix

\[ P_B = \sum_{\mu} \rho_{\mu\mu} \langle \mu | \mu \rangle \]

\[ P_T = \sum_{\mu} \rho_{\mu\mu}^T \langle \mu | \mu \rangle \]

these do not have to be diagonal but
\[ P_B = P_B^T \quad Tr P_B = 1 \]
\[ P_T = P_T^T \quad Tr P_T = 1 \]

If we sum over final states

\[ \frac{d\sigma}{d\Omega} = Tr(M P_B \otimes P_T M^T) \]

we could also detect certain
final spin states.

what is measured can be expressed in terms of some operator in
spin space

\[ \frac{d\sigma}{d\Omega} = Tr(\Theta M P_B \otimes P_T M^T) \]

It is conventional to measure the ratio of the cross sections
with \( \Theta = 0 \) in and off
\[ \langle \theta \rangle := \frac{\text{Tr}(\mathcal{O} M_{\rho^T \rho} M^T)}{\text{Tr}(M_{\rho^T \rho} M^T)} \]

The way this is usually treated is to generalize the treatment for spin \( \frac{1}{2} \).

\[ P = \text{polarization} := \frac{1}{2} (1 + \mathcal{P}) \]

most general 2x2 hermitian matrix with unit trace

If \( N = (2s+1)(2s'+1) \) there are \( N^2 \) independent hermitian matrices. Let \( S_i \) be the identity, \( \text{Tr} S_i = N \)

pick \( X_2, X_3 \) independent hermitian traceless matrices. Construct

\[ S_i = \mathcal{S}_i \]

\[ S_i S_j = N \delta_{ij} \]

\[ S_1 \]

\[ S_2 = X_2 \sqrt{N} / \text{Tr}(X_2) \]

\[ S_3 = Y_3 \sqrt{N} / \text{Tr}(Y_3) \]

\[ Y_3 = X_3 - S_2 \text{Tr}(S_2 X_1) / N \]
From we can expand
\[
\rho = \rho_n \mathbf{S}_n \quad \text{where} \quad \mathbf{S}_n = \mathbf{S}_m \mathbf{S}_n \mathbf{S}_m \mathbf{N}
\]
\[
\rho_n = \frac{1}{N} \text{Tr} (\rho \mathbf{S}_n)
\]
\[
\mathbf{O}_n = 2 \mathbf{O}_n \mathbf{S}_n \quad \alpha_n = \frac{1}{N} \text{Tr} (\rho \mathbf{O}_n)
\]

Then any spin observable can be constructed from
\[
K_{mn} = \text{Tr} (\mathbf{S}_m \mathbf{M} \mathbf{S}_n \mathbf{M}^+)
\]
\[
\langle \mathbf{O} \rangle = \frac{\sum \alpha_n \rho_m K_{mn}}{\sum K_{mn} \rho_m}
\]

From a theory point of view, everything can be calculated from \(K_{mn}\).

Identical particles

If the particles are identical, the detection does not distinguish the particles - the detection
responds to both (ca)

\[ \frac{d\sigma}{d\omega} = | F_{\mu\nu} \omega_{\mu\nu} (\vec{k}) \pm F_{\mu\nu} \omega_{\mu\nu} (-\vec{k})|^2 \]

where + is for bosons; - is for fermions (not we have exchanged me spins)