Last time

\[ H = H_0 + V_1 + V_2 \]

\[ H |\psi_1\rangle = \varepsilon_1 |\psi_1\rangle \]

\[ S_e = -4\pi^2\mu_0 \mu \frac{\langle k | T \mid k \rangle_e}{Z - E_1 - \Delta + i\eta/2} \]

\[ = -4\pi^2\mu_0 \mu \frac{\langle k | V_1 | \psi_1 \rangle\langle \psi_1 | V_2 | k \rangle_e}{Z - E_1 - \Delta + \frac{\eta}{2}} \]

\[ = -4\pi^2\mu_0 \mu \frac{1}{\pi\mu_0 k} \frac{\eta/2}{(Z - E_1 - \Delta)^2 + \eta/2} \]

\[ = \frac{4\pi\hbar}{R} \frac{\eta/2}{(Z - E_1 - \Delta)^2 + \eta/2} \]

\[ \sigma_e = \frac{2\pi + 1}{4\pi} \left( \frac{4\pi\hbar}{R} \right)^2 \frac{(\eta/2)^2}{(Z - E_1 - \Delta)^2 + \eta/4} \]

\[ |\psi\rangle \sim e^{-iT} |\psi\rangle \quad \text{with} \quad T = \frac{2\hbar}{\eta} \]

Lifetime of resonance

Relation to phase shift

\[ S_e = \frac{4\pi\hbar}{R} e^{i\delta_e} \sin S_e = \frac{4\pi\hbar}{R} \frac{\eta/2}{(Z - E_1 - \Delta)^2 + \eta/2} \]

\[ = \frac{4\pi\hbar}{R} \frac{\sin S_e}{e^{-i\delta_e}} = \frac{4\pi\hbar}{R} \frac{\sin S_e}{\cos S_e - i\sin S_e} \]

\[ = \frac{4\pi\hbar}{R} \frac{1}{\cos S_e - i} \]
This will be maximal when
\[ \cot \phi_0(E) = 0 \]
\[ \cot \phi_0(E) = 0 + \frac{d \cot \phi_0(E)}{dE} (E-E^*) + \ldots \]

has negative slope at \( 0 \)

Define \( -\frac{2}{\pi} = \frac{d \cot \phi_0(E)}{dE} \)

\[ \phi_0 \rightarrow \frac{4\pi \hbar}{k} \cdot \frac{1}{-\frac{2}{\pi} (E-E^*) - i} = \]

\[ \frac{4\pi \hbar}{k} \cdot \frac{\pi/2}{(E-E^*) + i \pi/2} \]

We see that the Breit-Wigner term of the scattering amplitude appears at energies \( E^* \) where \( \cot \phi_0 \rightarrow 0 \); i.e., at this energy \( \pi/2 \) is related to the derivative of the \( \cot \phi_0 \) by

\[ -\frac{2}{\pi} = \frac{d}{dE} \cot \phi_0(E) \bigg|_{E=E^*} \]
* Experimentally resonances are seen as a bump in the cross section with with $\Gamma$ at half max (in energy).

* $\frac{2\hbar}{\Gamma} \sim T$ lifetime of resonant state

* Theoretically $\cot S_0(E) = 0$ is the resonant energy $\gamma = \frac{\hbar}{\Delta E} \cot S_0(E)$ $E = E_0$ gives the width

**Example: $s$ shell potential**

$$V = -\lambda R S(r-R)$$

$$\phi_{S} = - \frac{2\lambda}{\hbar^3} \int dE \,(k^2/\hbar^2) \, h_e(k,\rho/\hbar)$$

$$\langle k | r \rangle = \frac{4\pi}{(2\pi \hbar)^3} \, i^2 \int dE \,(k r/\hbar)$$

$$\phi_{S}(x) = \sin x / x \quad h_{e}(x) = -i \frac{x}{\pi} e^{ix}$$

**Let's calculate $\langle k | T | k \rangle$.**

$$\langle k | k \rangle_0 + \int g_{S}(r, r', E) \, S(r-R) \, (r-R) \, \langle k | k \rangle_0 \, r'^2 dr$$

$$\langle k | k \rangle_0 = \langle k | k \rangle_0 - g_{S}(r, r', E) \, \lambda \hbar^3 \, \langle k | k \rangle_0$$
solving for $\langle R_1 R \rangle$

\[
\langle R_1 R \rangle = \left(1 + g \langle P \rangle \right) R^3 = \langle R_1 \rangle \langle R \rangle \
\]

\[
\langle R_1 \rangle = \frac{\langle R_1 R \rangle}{1 + \alpha R^3 g \langle R \rangle} \]

\[
\langle R_1 L \rangle = \langle R_1 V \rangle \langle \langle R \rangle \rangle \
\]

\[
\langle R_1 \rangle \left(-\alpha R^3 \delta(r-R) \langle \langle R \rangle \rangle \right) = -\alpha R^3 \langle R_1 \rangle \langle R_1 \rangle \
\]

\[
\langle R_1 L \rangle = -\alpha R^3 \frac{\langle R_1 \rangle \langle R_1 \rangle}{1 + \alpha R^3 g \langle R \rangle} \
\]

\[
= -\alpha R^3 \frac{(4\pi)^2}{(2\pi R)^3} \frac{J_0 \left(k R / k_{R} \right) J_0 \left(k R / h \right)}{1 + \alpha R^3 \left(-2i \frac{\delta}{h^2} \right) J_0 \left(k R / h \right) N_0 \left(k R / h \right)} \
\]

\[
= -\frac{\alpha R^3 8 \pi^2 k^3 R k^2}{8 \pi^2 k^2 R k^2} \frac{\sin^2 (R/R)}{\sin^2 (k R / h)} \frac{1 - 2ik R / h^2 - \alpha R^3 (-i) \left(\frac{\delta}{h^2} \right) \sin \frac{k R}{h} e^{i k R / h}}{- \alpha R^2 \frac{2}{h K}} \
\]

multi numerator denominator \times \frac{k^2}{2u}

\[
\frac{k^2}{2u} \frac{\alpha R}{\pi MHz} = \frac{\alpha R}{MHz} \
\]

\[
\frac{k^2}{2u} \frac{\alpha R^2}{MHz^2} = \frac{\alpha R}{MHz} \
\]

\[
\frac{k^2}{2u} \frac{\alpha R}{MHz^2} = \frac{\alpha R}{MHz} \
\]

\[
= -\frac{1}{\sin^2 \left(\frac{k R}{h} \right)} \frac{\alpha R \sin^2 \left(\frac{k R}{h} \right)}{\frac{k^2}{2u} - \frac{\alpha R}{MHz} \sin \frac{k R}{h} \cos \frac{k R}{h} - i \frac{\alpha R}{MHz} \sin \frac{k R}{h} } \
\]

\[-\frac{i}{\mu m R} \frac{\lambda R^R}{\hbar} \sin \left( \frac{R^R}{\hbar} \right) \]

\[\frac{k^2}{2\mu} - \frac{\lambda R^R}{\hbar} \sin \left( \frac{R^R}{\hbar} \right) \cos \frac{R^R}{\hbar} - i \frac{\lambda R^R}{\hbar} \sin \left( \frac{R^R}{\hbar} \right) = 0\]

to find the resonant energy we solve for \( k^* \)

\[\frac{k^*}{2} = -\frac{\lambda R^R}{\hbar} \sin \left( \frac{k^* R^R}{\hbar} \right)\]

\[-\frac{\mu m R}{\mu} \frac{\gamma_1}{\hbar} \left( E - E^* \right) + i \frac{\gamma_1}{\hbar}\]

\[E^* = \frac{\lambda R^R}{\hbar} \sin \left( \frac{k^* R^R}{\hbar} \right) \cos \left( \frac{k^* R^R}{\hbar} \right)\]

This only looks resonant when \( R \) is small.
Identical Particles

Consider a 2 particle system with particle 1 in state $|a\rangle_1$ and particle 2 in state $|b\rangle_2$ ($1,2$ identical)

The state of the system is 

$|a\rangle_1 |b\rangle_2$

In terms of quantum measurement, this state is indistinguishable from 

$|b\rangle_1 |a\rangle_2$

If $\langle a|b \rangle = 0$ then

$|a\rangle_1 |b\rangle_2 \perp |b\rangle_1 |a\rangle_2$

even though these states are physically indistinguishable, they are orthogonal.

It is useful to define an operation $P_{12}$ that interchanges the states of particle 1, 2
\[ P_{12} |a>_{1} |b>_{2} = |b>_{1} |a>_{2} \]

obviously

\[ P_{12}^2 |a>_{1} |b>_{2} = P_{12} |b>_{1} |a>_{2} = |a>_{1} |b>_{2} \]

\[ P_{12}^2 = I \]

consider an operator that acts only on particle 1 - i.e. the position \( x_1 \) of particle 1

\[ X_1 |x>_{1} |y>_{2} = x |x>_{1} |y>_{2} \]

\[ P_{12} X_1 P_{12} |x>_{1} |y>_{2} = P_{12} |x>_{1} |y>_{2} |x>_{2} \]

\[ = P_{12} |y>_{1} |x>_{2} = |y>_{1} |x>_{2} \]

but

\[ X_2 |x>_{1} |y>_{2} = y |x>_{1} |y>_{2} \]

comparing these expressions we see

\[ P_{12} X_1 P_{12} = X_2 \]

on mu generally

\[ P_{12} A_1 P_{12} = A_2 \]
Let's consider the Hamiltonian for a 2-electron atom

\[ H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_1 - r_2) + V(r_1) + V(r_2) \]

\[ P_{12} H P_{12} = \frac{P_{12} P_1 P_{12} P_1 P_{12} P_1 P_{12} P_1 P_{12} P_1}{2m} + \frac{P_{12} P_2 P_{12} P_2 P_{12} P_2 P_{12} P_2 P_{12} P_2}{2m} \]

\[ + P_{12} V(r_1 - r_2) P_{12} + P_{12} V(r_2) P_{12} + P_{12} V(r_2) P_{12} \]

\[ = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_2 - r_1) + V(r_2) + V(r_1) \]

Since \( V(r_2 - r_1) = V(r_1 - r_2) \) for the Coulomb interaction we have immediately

\[ P_{12} H P_{12} = H \]

\[ P_{12} H P_{12}^2 = H P_{12} = P_{12} H \]

or

\[ [P_{12}, H] = 0 \]

This means that we can find simultaneous eigenstates of \( H \) and \( P_{12} \).
since \((P_{12}^2 - I) = 0\), \(P_{12}\) has eigenvalues 
±1.

Define
\[
S = \frac{1}{2} (1 + P_{12})
\]
\[
A = \frac{1}{2} (1 - P_{12})
\]
\[
P_{12} S \Psi = \frac{1}{2} (P_{12} + P_{12}^2) \Psi = \frac{1}{2} (P_{12} + 1) \Psi = S \Psi
\]
\[
P_{12} A \Psi = \frac{1}{2} (P_{12} - P_{12}^2) \Psi = \frac{1}{2} (P_{12} - 1) \Psi = -A \Psi.
\]

Since
\[
H(\Psi) = E(\Psi)
\]
\[
P_{12} H \Psi = E(P_{12} \Psi)
\]
\[
H \Psi = E(P_{12} \Psi).
\]

If \(|\Psi\rangle\) is an eigenstate of \(H\) then
\[
S|\Psi\rangle, A|\Psi\rangle
\]
are simultaneous eigenstates of \(H\) and \(P_{12}\).

Note
\[
A^2 = \frac{1}{4} (1 - P_{12})(1 - P_{12}) = \frac{1}{4} (1 + 1 - 2 P_{12}) = \frac{1}{2} (1 - P_{12}) = A
\]
\[
S^2 = \frac{1}{2} (1 + P_{12})(1 + P_{12}) = \frac{1}{4} (1 + 1 + 2 P_{12}) = \frac{1}{2} (1 + P_{12}) = S
\]
we also have
\[ A \cdot S = \frac{1}{4} (1 - P_{1 \uparrow})(1 + P_{1 \uparrow}) = \frac{1}{4} (1 - 1 + P_{2 \uparrow} - P_{1 \uparrow}) = 0 \]

This shows that the space of eigenstates of \( H \) breaks up into 2 orthogonal subspaces that are either symmetric or antisymmetric under interchange of the electrons.

We have two problems:
(1) a physical state
(2) represented by 2 orthogonal vectors.

Not having a 1-1 correspondence between states and vectors would cause us to rethink our interpretation of quantum mechanics.

Nature's way out of this problem is to discard one of the vectors.
In order to better appreciate this problem consider a system of 3 identical particles. The states:

\[ |a\rangle_1 |b\rangle_2 |c\rangle_3 \]
\[ |a\rangle_1 |c\rangle_2 |b\rangle_3 \]
\[ |b\rangle_1 |c\rangle_2 |a\rangle_3 \]
\[ |b\rangle_1 |a\rangle_2 |c\rangle_3 \]
\[ |c\rangle_1 |b\rangle_2 |a\rangle_3 \]
\[ |c\rangle_1 |a\rangle_2 |b\rangle_3 \]

Here we have 1 physical state represented by six independent vectors.

def:

bosons - particles with integer spin
fermions - particles with half integral spin

bosons: \( \pi \) meson, \( \sigma \) particle
fermions: e electron, p proton
Nature's way of recovering quantum mechanics (i.e., each physical state represented by a vector up to normalization) is given by the symmetrization postulate:

1. States of \( n \) identical bosons are completely symmetric under interchange of identical particles.

2. States of \( n \) identical fermions are completely antisymmetric under interchange of identical particles.

* This gives \( 1 \) state \( \sim 1 \) vector.

* This postulate is a theorem that follows from the axioms of quantum field theory, i.e., that the field be local.
as far as we know there are
no exceptions to this postulate
for out 3 particle state the
unique vector for 

\[
\frac{1}{\sqrt{3}} \left( \langle a\rangle_1 \langle b\rangle_2 \langle c\rangle_3 + \langle a\rangle_1 \langle c\rangle_2 \langle b\rangle_3 + \\
\langle b\rangle_1 \langle a\rangle_2 \langle c\rangle_3 + \langle b\rangle_1 \langle c\rangle_2 \langle a\rangle_3 + \\
\langle c\rangle_1 \langle a\rangle_2 \langle b\rangle_3 + \langle c\rangle_1 \langle b\rangle_2 \langle a\rangle_3 \right)
\]

for fermions

\[
\frac{1}{\sqrt{3}} \left( \langle a\rangle_1 \langle b\rangle_2 \langle c\rangle_3 - \langle a\rangle_1 \langle c\rangle_2 \langle b\rangle_3 - \\
\langle b\rangle_1 \langle a\rangle_2 \langle c\rangle_3 + \langle b\rangle_1 \langle c\rangle_2 \langle a\rangle_3 + \\
\langle c\rangle_1 \langle a\rangle_2 \langle b\rangle_3 - \langle c\rangle_1 \langle b\rangle_2 \langle a\rangle_3 \right).
\]