Lecture 37

Special Relativity and Quantum Mechanics

Inertial coordinate system

* In the absence of forces free particles move with constant velocity

* Observation - the speed of light appears to be the same in many coordinate systems

* \( x^u = (ct, \vec{x}) \) = spacetime coordinate of an event

* \( d_{ab}^2 = (\vec{x}_a - \vec{x}_b)^2 - c^2 (t_a - t_b)^2 \)

* \( t_{ab} = \frac{1}{c^2} d_{ab}^2 \)

\( d_{ab} = \) proper distance between events \( a, b \)

\( t_{ab} = \) proper time between events \( a, b \)

\( c = \) speed of light
Transformations that preserve the proper time or proper distance between events are called Poincaré transformations.

\[ X^{'\mu} \rightarrow X^{'\mu} = f^\mu(X) \]
\[ X^{'\mu}_b \rightarrow X^{'\mu}_b = f^\mu(X_b) \]

\[ (X_a - X_b)^2 = \eta_{\mu\nu}(X_a - X_b)^\mu (X_a - X_b)^\nu = d_{ab} = \]

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ d_{ab} = \eta_{\mu\nu}(f^\mu(X_a) - f^\mu(X_b))(f^\nu(X_a) - f^\nu(X_b)) \]

The most general transformation of the form \( X^\mu \rightarrow X^{'\mu} = f^\mu(x) \) has the form

\[ X^{'\mu} = \Lambda^\mu_\nu x^\nu + a^\mu \]

\[ \eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu \]

where \( \Lambda^\alpha_\beta \) is a constant matrix called a Lorentz transformation and \( a^\mu \) is a constant 4 vector.
I leave the complete proof as a homework exercise, but

\[(x - x_v)^2 = (f(x_o) - f(x_v))^2\]

\[
\begin{align*}
\frac{2}{\partial x_u} & \quad \text{set } x_u = 0 \quad = 0 \\
\frac{2}{\partial x_v} & \quad \text{set } x_v = 0 \\
\end{align*}
\]

\[
\frac{\partial f''(u)}{\partial x_v}(0) = A_v^{u}, \quad a^u = f''(0)
\]

Principle of Special Relativity

(i) Any two inertial coordinate systems are related by Poincaré transformation

(ii) Equivalent experiments performed in different inertial coordinate systems give equivalent results.
\[ \eta_{\mu \nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha \beta} \]

\[ (\Lambda^T)_{\mu}^\nu \eta_{\mu \nu} \Lambda^\nu_\beta = \eta_{\alpha \beta} \]

\[ \text{det} \Lambda^T \text{det} \eta \text{det} \Lambda = \text{det} \eta \]

\[ (\text{det} \Lambda)^2 = 1 \]

\[ \therefore \text{det} \Lambda = \pm 1 \]

\[ -\Lambda^0_0 \Lambda^0_0 + \Lambda^i_0 \Lambda^i_0 = -1 \]

\[ (\Lambda^0_0)^2 = 1 + \sum (\Lambda^i_0)^2 \]

The Lorentz transformations fall into 4 classes:

- \[ \text{det} \Lambda = 1 \quad \Lambda^0_0 \geq 1 \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Identity} \]
- \[ \text{det} \Lambda = 1 \quad \Lambda^0_0 \leq -1 \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Space-time reversal} \]
- \[ \text{det} \Lambda = -1 \quad \Lambda^0_0 \geq 1 \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Space reflection} \]
- \[ \text{det} \Lambda = -1 \quad \Lambda^0_0 \leq -1 \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Time reversal} \]

The transformations connected to the identity include 3 rotations and rotationless Lorentz boosts:

\[ \Lambda^\mu_\nu \rightarrow \begin{pmatrix} \nu^0 & \nu^i \\ \nu^i & \delta^i_j + \frac{\nu^i \nu_j}{1 + \nu^0 \nu_0} \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \nu^0 \\ \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix} \]
transformation properties

(1) quantities that transform like
\[ x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \]
are called 4 vectors

\[ x^\mu = \eta_{\mu \nu} x^\nu = \text{covariant components of 4 vecr} \]

\[ x_\mu x^\mu = \eta_{\mu \nu} x^\mu x^\nu \text{ is Lorentz invariant} \]

since it is essentially a proper (distance)²

\[ x_\mu \rightarrow x'_\mu = \eta_{\mu \nu} x'^\nu = \eta_{\mu \nu} \Lambda^\nu_\sigma x^\sigma \]
\[ = \underbrace{\eta_{\mu \nu} \Lambda^\nu_\sigma} \eta^{\nu \rho} x_\rho \]
\[ = \Lambda^\rho_\mu x_\rho \]

\[ x_\mu \rightarrow x'_\mu = \Lambda^\rho_\mu x_\rho \]

where \( \eta^{\mu \nu} = \eta_{\mu \nu} \) \quad \eta^\mu_\nu = f^\mu_\nu = \eta^{\mu \kappa} \eta_{\kappa \nu} \]
the 2 types of components
\[ x^u = (ct, \bar{\vec{x}}) \]
\[ x^\nu = (-ct, \bar{\vec{x}}) \]
are used because one of each type of 4 vector is need to make an invariant.

we use \( \eta^{\mu\nu} n_{\mu\nu} \) to raise or lower indices.

Wigner: (1939) In a relativistic quantum theory the Poincaré group is a symmetry of the quantum theory.

This means
\[ (\lambda, a) \rightarrow U(\lambda, a) \]
\[ (\lambda_2 a_2)(\lambda_1 a_1) = (\lambda_2 \lambda_1, \lambda_2 a_1 + a_2) \]
\[ U(\lambda_2 a_2)U(\lambda_1 a_1) = U(\lambda_2 \lambda_1, \lambda_2 a_1 + a_2) \]
\[ U^+(\lambda a) = U(\lambda a). \]

\( U(\lambda a) \) is a unitary representation of the Poincaré group.
\[ |\psi\rangle \rightarrow |\psi'\rangle = U(\lambda \epsilon) |\psi\rangle \]
\[ |\phi\rangle \rightarrow |\phi'\rangle = U(\lambda \epsilon) |\phi\rangle \]
\[ \downarrow \]
\[ K \psi^\dagger |\psi\rangle^2 = K \psi'^\dagger |\phi'\rangle^2 \]

This shows that probability is the same in all inertial coordinate systems.

This characterization of symmetries was made 13 years after the first attempts to make a relativistic quantum theory.

Before that time,

1. solutions of classical equations are normally observable
2. changes of inertial coordinate system changes the equation and solution. If the equation is invariant (covariant) the solution is expected to be invariant or covariant.
This is how relativity is realized in classical wave equations.

* Klein-Gordon Schrödinger equation

(Schrödinger wrote this equation down between Klein and Gordon.

\( \bar{p} = \frac{k}{i} \frac{\partial}{\partial x} \)

\( H = i \hbar \frac{\partial}{\partial t} = -\frac{\hbar^2}{2m} \)

\( p^\mu = \left( \frac{E}{c}, p_x, p_y, p_z \right) \) 4 momentum

\( p^\mu p_\nu = \bar{p}^2 - \frac{E^2}{c^2} = -m^2 c^2 \)

Try

\( \left( -\hbar^2 \nabla^2 + \frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + m^2 c^2 \right) \psi = 0 \)

\( \left( -\Box - \frac{m^2 c^2}{\hbar^2} \right) \psi = 0 \)

\( \Box = \frac{\partial^2}{\partial x^0 \partial x^0} - \frac{\partial^2}{\partial x^1 \partial x^1} = \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} \)

This equation is called the Klein-Gordon equation. It attempts to replace the Schrödinger equation for a free particle.
it is easy to write plane
wave solution of the form
\[ e^{i p \cdot x / \hbar} \]
\[ \left( -\frac{\hbar^2}{2m} + \frac{\vec{p}^2}{2m} + \frac{m^2 c^4}{2m} \right) e^{i p \cdot x / \hbar} = 0 \implies \]
\[ p^0 = \pm \sqrt{\vec{p}^2 + m^2 c^2} = E / c \]

one problem with this equation
is that if we want a complete
set of solutions some of them
have negative energy.

a second problem has to do
with probability
\[ \frac{d}{dt} \int \psi^* \psi = \int \left( \frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} \right) = 0 \]

this is basically because \( \psi \)
satisfies an equation that is
second order in time.

one can try to fix this by changing
the definitiveness of the probability — but
the most natural choice gives
a probability that is not positive
definite
For these reasons — and because it was unable to explain the fine structure of hydrogen — it was eventually discarded.

Dirac wanted a covariant transforming equation that was first order in time

\[
\frac{i\hbar}{c} \frac{\partial \psi}{\partial t} = -\frac{i\hbar}{c} (\vec{\alpha} \cdot \vec{\nabla}) \psi + \beta mc^2 \psi = H \psi
\]

Dirac wanted this equation to transform covariantly with respect to Lorentz transformation

\[
\frac{\partial}{\partial x^\mu} = (\frac{1}{c} \frac{\partial}{\partial \tau}, \vec{\nabla})
\]

The covariance of \( \vec{\alpha} \) could not be simple scalar quantities.

Dirac assumed that \( \vec{\alpha} \beta \) were \( N \times N \) matrices.
\[ i \frac{\hbar}{c} \frac{\partial \psi_i}{\partial t} = -i \hbar \left( \alpha_{ij} \frac{\partial}{\partial x_j} + \alpha_{ij} \frac{\partial^2}{\partial x_j^2} + \alpha_{ij} \frac{\partial}{\partial x_j} \right) \psi_j + 2 \beta_{ij} m c^2 \psi_j \]

\[ = H_{ij} \psi_j \]

\[ i \frac{\hbar}{c} \frac{\partial}{\partial t} \left( i \frac{\hbar}{c} \frac{\partial}{\partial t} \psi_i \right) = H_{ij} \left( i \frac{\hbar}{c} \frac{\partial}{\partial t} \psi_j \right) \]

\[ = \sum_k H_{ij} H_{jk} \psi_k \]

\[- \frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} \psi_i = H_{ij} H_{jk} \psi_k \]

\[ \left( - \frac{\hbar^2 c^2}{2 \hbar^2} + m^2 c^2 \right) \delta_{ik} \psi_k \]

\[ \left[ -i \hbar \left( \alpha_{ij} \frac{\partial}{\partial x_j} + \alpha_{ij} \frac{\partial^2}{\partial x_j^2} + \alpha_{ij} \frac{\partial}{\partial x_j} \right) + \beta_{ij} m c^2 \right] \]

\[ \left[ \left( \alpha_{ij} \frac{\partial^2}{\partial x_j^2} + \alpha_{ij} \frac{\partial}{\partial x_j} + \alpha_{ij} \frac{\partial^2}{\partial x_j^2} + \alpha_{ij} \frac{\partial}{\partial x_j} \right) + \beta_{ij} m c^2 \right] \]

\[ \alpha_{ij} = \delta_{in} \alpha_{jk} = \delta_{in} \alpha_{jk} \]

\[ \alpha_{ij} + \alpha_{ij} = \delta_{in} \]

\[ \alpha_{ij} = \delta_{ik} \]

\[ \beta_{ij} = \delta_{jk} \]
If we suppress matrix indices we require
\[ \alpha^i \alpha^j + \alpha^j \alpha^i = 2 \delta^{ij} \]
\[ \alpha^i \beta + \beta \alpha^i = 0 \]
\[ \alpha^i \alpha^i = \beta^2 = 1 \]

We need 4 independent anticommuting Hermitian matrices
\[ \text{Tr} (\alpha^i) = \text{Tr} (\beta \alpha^i) = \text{Tr} (\beta \alpha^i \beta) = \text{Tr} (-\alpha^i \beta^2) = \text{Tr} (-\alpha^i) = -\text{Tr} (\alpha^i) = 0 \]
\[ (\alpha^i)^2 - 1 = 0 \]

Thus the eigenvalues are ±1; they come in pairs — since the trace is 0 there must be an equal number of positive and negative eigenvalues so \(\dim \delta\) is even.

\(N=2\) there are only 3 independent anticommuting Hermitian operators with 0 trace. (Pauli matrices)
The minimum size for the Dirac equation is \(N=4\).
One solution is
\[ \chi_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \]
\[ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \frac{i\hbar}{c} \frac{\partial \Psi}{\partial t} = (\vec{a} \cdot \vec{p} + \beta mc^2) \Psi \]

This is called the free particle Dirac equation. Properties

\[ \frac{i\hbar}{c} \frac{\partial}{\partial t} (\psi^+ \psi) = \frac{i\hbar}{c} \left( \frac{\partial \psi^+}{\partial t} \psi + \psi^+ \frac{\partial \psi}{\partial t} \right) \]

\[ = - (\hat{H} \psi^+) \psi + \psi^+ \hat{H} \psi \]

\[ = - (\vec{a} \cdot \nabla \psi^+) \psi + \psi^+ \vec{a} \cdot \nabla \psi \]

\[ = \nabla \left( \psi^+ \vec{a} \psi \right) \]

\[ \frac{d}{dt} \int \psi^+ \psi = \int \frac{c}{i \hbar} \nabla (\psi^+ \vec{a} \psi) = \frac{c}{i \hbar} \int \nabla \cdot (\psi^+ \vec{a} \psi) = 0 \]

The Dirac equation has a conserved probability.