Identical Particles

Consider a system consisting of 2 identical particles.

Assume particle 1 is in state $a$ and particle 2 is in state $b$; $\langle a | b \rangle = 0$.

Then $|a\rangle_1 |b\rangle_2 = |b\rangle_2 |a\rangle_1$,

(1) represent the same physical state

(2) the vectors are independent and orthogonal.

There is no longer a 1-1 correspondence between rays (vectors up to a complex multiple) and states.

Define the transposition operator $P_{12}$ by

$P_{12} |a\rangle_1 |b\rangle_2 = |b\rangle_1 |a\rangle_2$

$P_{12}^2 |a\rangle_1 |b\rangle_2 = P_{12} |b\rangle_1 |a\rangle_2 = |a\rangle_1 |b\rangle_2$

$\langle c | d \rangle P_{12} |a\rangle \langle b | = \langle c | d | b\rangle_2 |a\rangle_2^* = \langle c | d | b\rangle_2 |a\rangle_2^* = \langle c | d | b\rangle_2 |a\rangle_2^*$
\[
\langle c|b\rangle^* \langle d|a\rangle = \langle b|c\rangle \langle a|d\rangle = \langle b|c\rangle \langle a|d\rangle.
\]
\[
\langle a|b|1d\rangle \langle 1c\rangle = \langle a|b|1d\rangle \langle 1c\rangle.
\]
Thus \[ P_{12}^2 = I, \quad P_{12}^* = P_{12}. \]

Let \[ A_1 = A_1^+ \] act on the Hilbert space of particle 1.

\[
A_1 \langle a\rangle_1 = a \langle a\rangle_1
\]
\[
A_1 \langle a\rangle_2 = a \langle a\rangle_2
\]

\[
P_{12} A_1 P_{12} \langle a\rangle_1, \langle a\rangle_2 = P_{12} A_1 \langle a\rangle_1, \langle a\rangle_2 = a^* P_{12} \langle a\rangle_1, \langle a\rangle_2 = a^* \langle a\rangle_1, \langle a\rangle_2
\]

\[
P_{12} \langle a\rangle_1, \langle a\rangle_2 = a^* \langle a\rangle_1, \langle a\rangle_2
\]

Comparing these expressions we see that

\[
A_2 = P_{12} A_1 P_{12}
\]

\[
P_{12} \left( H_1 + H_2 + V_{12} \right) P_{12} = H_2 + H_1 + V_{21}
\]

for identical particle \[ V_{12} = V_{21} \]

\[
P_{12} H P_{12} = H
\]

\[
\left[ P_{12} H \right] = 0
\]
This means that it is possible to find simultaneous eigenstates of $H$ and $P_{12}$.

To find eigenstates of $P_{12}$ note $P_{12}^2 = I = 0$ implies eigenvalues are $\pm 1$.

$$S = \frac{1}{2} (I + P_{12})$$

$$A = \frac{1}{2} (I - P_{12})$$

$$P_{12} S \psi = S P_{12} \psi$$

$$P_{12} A \psi = -A P_{12} \psi$$

If $\psi$ is an eigenstate of $H$

$$H \psi = E \psi$$

$$H S \psi = S H \psi = E S \psi$$

$$H A \psi = A H \psi = E A \psi$$

(Since $\{ H, P_{12} \} = \{ H I I I = 0 \}$).

Thus $H$ has eigenstates that are symmetric and antisymmetric under interchange of identical particles.
In order to preserve the 1-1 correspondence between states and Hilbert space vectors, Pauli introduced the symmetry spin postulate:

The state vector for a system of N identical particles must be

(a) symmetric under interchange of particles in particles with integer spin (bosons)

(b) antisymmetric under interchange of particles in particles with half integer spin (fermions)

This assumption can be derived by requiring in quantum field theory that physical observables in non causally connected space time regions are independent.
This postulate explains the shell structure of atoms — no 2 electrons can be in the same state (spin, orbital, n).

When we have N identical particles there are N! vectors representing the same physical state: the symmetrization principle reduces this to one state.

Consider \( N = 3 \)

\[ \frac{1}{\sqrt{3}} \left( |a\rangle |b\rangle |c\rangle + |a\rangle |c\rangle |b\rangle + |b\rangle |a\rangle |c\rangle + |b\rangle |c\rangle |a\rangle + |c\rangle |a\rangle |b\rangle + |c\rangle |b\rangle |a\rangle \right) \]

is the symmetrized (antisymmetrized) state representing the physical state.
Note that if \( |a\rangle \langle b| \langle c| \) are all orthogonal the norm of this state has 36 terms; six of which are 1 and the rest vanish. The factor \( \sqrt{3} \) ensures unit normalization in this case.

We can construct the analog of \( A_i \) for \( N \) particle states. We need a definition:

1. A permutation \( \sigma \) of \( N \) objects is a function \( \sigma(i) = j \)

\[
i = 1 \ 2 \ 3 \ \ldots \ \ N
j = \sigma(1) \ \sigma(2) \ \sigma(N)
\]

where \( \{ \sigma(i) \} = \{ 1 \ldots N \} \).

For \( N = 3 \):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}
\]

there are \( 6 = 3! \) permutations in \( N = 3 \):

\[
P_{12}^2 \ \ P_{23} P_{31} \ \ P_{13} P_{22}
\]

\[
P_{23} \ \ P_{13} \ \ P_{12}
\]
we see that some of the permutation are products of even # of transpositions while some are products of odd number of transpositions.

let \( 16! = \sum_{\sigma} \sigma \) can be expressed as an odd # of transpositions.

let \( 16! = \sum_{\sigma} \sigma \) can be expressed as an even # of transpositions.

we also note

1. \( P_{21} P_{31} \neq P_{31} P_{21} \)
2. \( (P_{21} P_{31})^+ = P_{31} P_{21} \)

we conclude that the order of transpositions matters.

\[
S = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}(N)} \sigma P_0
\]

\[
A = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}(N)} (-1)^{\ell(\sigma)} \sigma P_0
\]

are Hermitian

\[
S^2 = \frac{1}{N'^2 N!} \sum_{\sigma, \sigma'} \sigma \sigma' P_0 P_0' = \frac{1}{N'^2 N!} \sum_{\sigma, \sigma'} \sigma P_0 \sigma' P_0'
\]

where \( \sigma \sigma'(i) = \sigma(\sigma'(i)) \).

In fixed \( \sigma \) \( \sum_{\sigma'} \sigma' P_0 = \sum_{\sigma'} \sigma' P_0' \) (HW trim \( N = 3 \))
\[ S^2 = \frac{1}{N!N!} \sum_{\sigma_1} P_{\sigma_1} = S \]

Similarly

\[ A^2 = \frac{1}{N!N!} \sum_{\sigma_1} (-1)^{\sigma_1^{161} - \sigma_1^{161}} P_{\sigma_1} \]

For fixed \[ \sigma \]

\[ \sum (-1)^{\sigma^{161} + \sigma^{161}} P_{\sigma} = (-1)^{\sigma^{161}} P_{\sigma} \]

\[ A^2 = \frac{N!}{N!N!} \sum_{\sigma_1} (-1)^{\sigma_1^{161}} P_{\sigma} \]

One can also show

\[ A \cdot S = 0 \quad (HU) \]

Physical eigenstates of a Hamiltonian with \[ N \]
identical particles are eigenstates of \[ A \] (fermions)
\[ \sigma \] is (bosons).

\[ \mid \Psi \rangle \rightarrow \mid \Psi \rangle_{\text{space}} \times \mid \Psi \rangle_{\text{spin}} \]

When the symmetry postulate is applied to this system, the total wave function has to be symmetric or antisymmetric.
For $A \Psi = \Psi$

we can have

$\Psi$ space symmetric
$\Psi$ spin antisymmetric

For $S \Psi = \Psi$

we can have

$\Psi$ space symmetric
$\Psi$ spin symmetric
$\Psi$ space antisymmetric
$\Psi$ spin antisymmetric

In treating systems of $N$ identical particles it is useful to introduce a special basis with the desired symmetries.
Let \( |\psi\rangle \) be any orthonormal 1 particle basis.

Define \( |\rho\rangle \) a particle state \( \psi \) we normalize this state with \( \langle \rho | \rho \rangle = 1 \)

We define

\[ |n_1 n_2 \ldots n_\omega \rangle \]

\( n_1 \) particles in state 1
\( n_2 \) particles in state 2
\( n_3 \) particles in state 3

\( \langle n_\omega \ldots n_1 | n_1 \ldots n_\omega \rangle = \delta_{n_\omega n_\omega} \delta_{n_1 n_1} \ldots \)

We introduce an algebra of operators on these states

\[ a_i^\dagger |n_1 \ldots n_i \ldots n_\omega \rangle = \]

\[ c_i |n_1 \ldots n_i+1 \ldots n_\omega \rangle \]

\[ a_i^\dagger |n_i \rangle = c_i |n_i+1 \rangle \]

\[ \langle m_\omega | a_i | n_i \rangle = \langle n_i | a_i^\dagger | m_i \rangle^* = c_i^* \langle n_i | m_i+1 \rangle^* = c_i^* \langle m_i+1 | n_i \rangle = c_i \langle m_i | n_i-1 \rangle. \]

\[ a_i^\dagger |n_i \ldots n_i \ldots n_\omega \rangle = |n_1 \ldots n_{i-1} \ldots n_\omega \rangle c \]
For identical bosons, we require
\[ a_i a_j = a_j a_i; \]

For identical fermions, we require
\[ a_i a_j = -a_j a_i; \]

we choose the normalization so
\[ a_i a_i^+ + a_i^+ a_i = \delta_{ij}; \]

the condition \( a_i a_j + a_j a_i = 0 \) means that 2 fermions cannot occupy the same state.

The normalization of the operators determines the normalization of the states.

\[ (a_i^+)^n \left| 0 \right> = \left| 0 \right> \]
\[ \langle 0 | (a_i^+)^n (a_i^+)^n | 0 \rangle = n! \langle 0 | 0 \rangle \]

\[ |n_i\rangle = \frac{1}{\sqrt{n_i!}} (a_i^+)^n |0\rangle \]

\[ a_i^+ |n_i\rangle = \frac{1}{\sqrt{n_i!}} (a_i^+)^{n+1} |0\rangle = \sqrt{n_i+1} \frac{1}{\sqrt{n_i!+1}} (a_i^+)^{n+1} |0\rangle = \sqrt{n_i+1} |n_i+1\rangle \]

\[ a_i |n_i\rangle = \frac{1}{\sqrt{n_i!}} a (a_i^+)^n |0\rangle = \frac{n}{\sqrt{n_i!}} (a_i^+)^{n-1} |0\rangle = \sqrt{n} \frac{1}{\sqrt{n_i+1}} (a_i^+)^{n-1} |0\rangle = \sqrt{n} |n_i-1\rangle \]
For fermions \( n_i \) can only be 0 or 1; however, we need to define a phase convention.

\[ \langle n_i \rangle \quad \langle n_n^w \rangle = (a_i^+)^n (a_i^+)^{n^w} - (a_i^+)^{n^w} \langle 0 \rangle \]

\[ a_j^+ \langle n_i \rangle \quad \langle n_n^w \rangle = (-)^{n_i} \langle n_i \rangle \quad \langle n_i+1 \rangle \quad \langle n_n^w \rangle \]

\[ a_j \langle n_i \rangle \quad \langle n_n^w \rangle = (-)^{n_i} \langle n_i \rangle \quad \langle n_i-1 \rangle \quad \langle n_n^w \rangle \]

In these expressions \( n_i \in \{0,1\} \).

All of these expressions are with respect to a given single particle basis.

\[ \{ \langle n_i \rangle_a \} \quad \{ \langle n_i \rangle_b \} \]

\[ \langle n_i \rangle_a = \frac{\xi}{m} \quad \langle n_i \rangle_b \quad \langle m 1 i \rangle_a \]

\[ a_{ia}^+ |0\rangle = \langle i |_a = \xi \quad a_{mb}^+ |0\rangle = \langle m 1 i \rangle_a \]

\[ a_{ia} = \frac{\xi}{m} \quad a_{mb}^+ \quad \langle m 1 i \rangle_a \]

\[ a_{ia} = \frac{\xi}{m} \quad \langle i | \rangle \quad a_{mb} \]
assume \[ \left[ a_{mn}^t a_{nb}^t \right]_z = \delta_{m+n}^k \]

\[ \left[ a_{ia} a_{jb}^t \right]_z = \frac{2}{m,n} \langle i | m \rangle_b \left[ a_{mn}^t a_{nb}^t \right]_z \langle n | j \rangle_a \]

\[ = \frac{2}{m,n} \langle i | m \rangle_b \delta_{mn} \langle n | j \rangle_a \]

\[ = \frac{2}{m} \langle i | m \rangle_b \langle m | j \rangle_a \]

\[ = \langle i | j \rangle_a = \delta_{ij} \]

we see that the algebraic relations are independent of the choice of basis.