Lecture 38

\[ x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + \alpha^\mu \]

\[ \eta_{\mu\nu} \Lambda^\mu_\sigma \Lambda^\sigma_\beta = \eta_{\alpha\beta} \]

properties of \( \Lambda \)

\[ \Lambda^T \eta \Lambda = \eta \]

\[ \det \eta = \det (\Lambda^T \eta \Lambda) = \det \Lambda^T \det \eta \det \Lambda = \det \Lambda \det \eta \det \Lambda \]

\[ (\det \Lambda)^2 = 1 \]

since \( \lambda^\mu_\nu \) is real

\[ \det \Lambda = \pm 1 \]

also

\[ \eta_{\mu\nu} = \eta_{\mu\nu} \lambda^\mu_\rho \lambda^\rho_\nu + \eta_{\mu\nu} \lambda^\mu_\rho \lambda^\rho_\nu \]

\[ -1 = - (\lambda^\mu_\rho)^2 + 2 (\lambda^\mu_\rho)^2 \]

\[ (\lambda^\mu_\rho)^2 = 1 + 2 (\lambda^\mu_\rho)^2 \geq 1 \]

\[ \lambda^\mu_\rho \geq 1 \quad \text{or} \quad \lambda^\mu_\rho \leq -1 \]

we can classify Lorentz transformations as

\[ \det \Lambda = 1 \quad \lambda^\mu_\rho \geq 1 \quad \text{includes} \quad I \]

\[ \det \Lambda = 1 \quad \lambda^\mu_\rho \leq -1 \quad \text{includes} \quad T \rho = -(I) \]

\[ \det \Lambda = -1 \quad \lambda^\mu_\rho \geq 1 \quad \text{includes} \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \det \Lambda = -1 \quad \lambda^\mu_\rho \leq -1 \quad \text{includes} \quad T \rho = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
since the weak interaction violates the discrete symmetries → relativity is only associated with the subgroup with det $\Lambda = 1$ $\Lambda^\dagger \Lambda = 1$ This includes the identity

These $\Lambda$ include

(1) Rotations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R^\dagger R = 1 \quad \text{det} R = 1$$

(2) Rotationsless LT

$$\tilde{v}^0 = \sqrt{1 - \tilde{v}^2}$$

$$\Lambda(\tilde{v}) = \begin{pmatrix} \tilde{v}^0 & \tilde{v} \\ \tilde{v} & \delta_{ij} + \frac{\tilde{v}^0 \tilde{v}_i}{1 + \tilde{v}^0} \end{pmatrix}$$

$$\Lambda(\tilde{v}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{v}^0 \\ \tilde{v} \end{pmatrix}$$

Some 4 vectors

$$X^\mu$$

$$\tilde{v}^\mu = \frac{dx^\mu}{d\gamma} = \lim_{\gamma \to 0} \frac{X^\mu(\gamma + \Delta \gamma) - X^\mu(\gamma)}{\Delta \gamma}$$

since the derivative involves a difference of coordinates

$$\tilde{v}^\mu \to \tilde{v}^\mu = \Lambda^\mu_\nu \tilde{v}^\nu \quad \text{(no translation)}$$
\( y^{-4} \text{ velocity} \)

\[
y^{-4} = \begin{pmatrix} \frac{dt}{ds} \\ \frac{d\bar{v}}{ds} \end{pmatrix}
\]

\[
(d\tau)^2 = (dt^2) - \frac{1}{c^2} (dx)^2
\]

\[
(d\tau^2) = (ds)^2 + \frac{1}{c^2} (d\bar{x})^2
\]

\[
1 = (d\tau^2)^2 = \frac{1}{c^2} (d\bar{x})^2
\]

\[
\sqrt{1 - \frac{v^2}{c^2}} = \frac{d\tau}{dt}
\]

\[
\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma
\]

\[
\gamma^{-4} = \begin{pmatrix} c\gamma \\ \bar{v}\gamma \end{pmatrix}
\]

\[
\gamma^{-4} = \begin{pmatrix} c \frac{dr}{dt} \\ \frac{d\bar{v}}{dt} \end{pmatrix} = \begin{pmatrix} c \frac{dr}{dt} \\ \frac{d\bar{v}}{dt} \end{pmatrix}
\]

\[
\frac{dy}{dt} = -\frac{1}{2} \frac{1}{r^3} \left( -\frac{2\bar{v}\cdot\bar{a}}{c^2} \right) = \frac{1}{r^3} \frac{\bar{v} \cdot \bar{a}}{c^2}
\]

\[
\gamma^{-4} = \begin{pmatrix} \frac{1}{r^2} \frac{\bar{v} \cdot \bar{a}}{c^2} \\ \bar{a} \frac{1}{r^2} \frac{\bar{v} \cdot \bar{a}}{c^2} \end{pmatrix}
\]

**In the particles rest frame** \( y \to 1, \bar{v} \to 0 \)

\[
\gamma^{-4} \to \begin{pmatrix} 0 \\ \bar{a} \end{pmatrix}
\]
Newtons laws are generalized so
\[ ma^u = \left( \begin{array}{c} 0 \\ 0 \\ \end{array} \right) \] in the $\bar{v}=0$ frame

\( \left( \begin{array}{c} 0 \\ F \end{array} \right) \) is assumed to transform as a 4 vector - in a frame where

the particle moves with velocity $\bar{v}$

\[ d^u = \left( \begin{array}{c} \bar{v} \\ \bar{v}^2 I + \frac{\bar{v} \bar{v}}{1+\bar{v}^2} \end{array} \right) (\begin{array}{c} 0 \\ 0 \end{array}) = \left( \begin{array}{c} \bar{v} \cdot F \\ 1+\bar{v} \cdot \bar{v} \end{array} \right) \] $\vec{v} \rightarrow \frac{\bar{v}}{c}$

when $\bar{v}=0, d^u = 0 \Rightarrow$ in the absence of force

\[ m \frac{d\bar{v}}{ds} = \frac{d}{ds}(mv^u) = \frac{d}{dt}(p^u) = 0 \]

\[ p^u = mv^u \]

is a 4 vector that is conserved in the absence of forces $\Rightarrow$

the 0 component is associated with the energy - the vector components are associated with the linear momentum.
\( P^\mu P_\mu = m^2 \bar{v}^2 - m^2 (c^2 - \bar{v}^2) = -m^2 c^2 \)
\[ = -(\bar{P}^\mu)^2 + (\bar{P})^2 = -m^2 c^2 \]
\((E/c)^2 = \bar{P}^2 + m^2 c^2 \)
\[ \bar{E} = \sqrt{\bar{P}^2 c^2 + m^2 c^4} \]

Thus, \( P^\mu a^\mu, u^\mu, x^\mu \) are all 4 vectors.

Quantum Theory

\[
(\lambda_2 a_2)(\lambda_1 a_1) = (\lambda_2 \lambda_1 a_1, \lambda_2 a_1 + a_2)
\]

Relativity = Poincare group
\[ \text{a symmetry group of a quantum theory} \]

\[
U(\lambda_2 a_2) U(\lambda_1 a_1) = U(\lambda_2 \lambda_1 a_1, \lambda_2 a_1 + a_2)
\]

\[
|\psi'\rangle = U(\lambda \xi) |\psi\rangle
\]
\[
|\phi'\rangle = U(\lambda \xi) |\phi\rangle
\]

\[
P' = |\langle \psi' | \phi' \rangle|^2 = K \psi(U^\dagger U |\phi\rangle^2 = K |\psi |^2 = P
\]

Experiments cannot distinguish these
In this case the problem is to construct \( \mathcal{U}(\Lambda a) \). This is not entirely trivial because \( \mathcal{U}(I, (t,0,0,0)) = e^{-iHt/\hbar} \) involves the Hamiltonian.

Time evolution can also be expressed using combinations of Lorentz transformations and translations — which puts interaction-dependent constraints on these interactions.

\[ m^2 c^4 = E^2 - p^2 c^2 \]
\[ E^2 = p^2 c^2 + m^2 c^4 \]

One possibility (in free particle case):
\[ E = i\hbar \frac{\partial}{\partial t} \]
\[ p = -i\hbar \vec{\nabla} \]

\[ i\hbar \frac{\partial^q}{\partial t} = \sqrt{-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4} \Psi \]
This is called the relativistic Schrödinger equation — it was discarded because
(1) the square root of $\Box$ was difficult to interpret
(2) very non-symmetric treatment of space and time.

(it is actually an acceptable equation based on Wigner's criteria)

Schrödinger + Klein Gordon use

$E^2 = p^2 c^2 + m^2 c^4 \Rightarrow$

$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$

This had the advantage that it has a very symmetric treatment of space and time

$\partial_{\mu} = \left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$\partial_{\mu} = \left( -\frac{\partial}{\partial \xi}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$\Box = -\partial_{\mu} \partial_{\mu}$

$(-\hbar^2 c^2 \Box - m^2 c^4) \psi = -$
\[(D^2 + \frac{m^2c^2}{\hbar^2}) \psi = 0\]

This is the standard form of the Klein Gordon equation. 
\(D^2 = -\partial^2\)

problems

1. Plane wave solutions \((x^0 = ct)\)

\[e^{ip \cdot x / \hbar}\]

\[-\frac{p^2}{\hbar^2} + \frac{\vec{p}^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2}\]

\[E^2 = p^2c^2 + m^2c^4\]

\[E = \pm \sqrt{\vec{p}^2c^2 + m^2c^4}\]

\[P_0 = \pm \sqrt{\vec{p}^2 + m^2c^2}\]

The immediate problem is that this system has eigenstates with arbitrarily negative energy.

2. \[\frac{d}{dt} \int \psi^* \psi = \int \left( \frac{\partial \psi^* \psi}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} \right) d^3x \neq 0\]

you can try to change the definition of \(P \rightarrow \) but the conserved quantity is not positive definite.
normally one has

\[\frac{dP}{dt} = \int \left( \frac{\partial}{\partial \bar{\psi}} \psi + \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \psi \right) \frac{d}{dt} \]

\[\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \mathcal{H} \psi \quad \frac{\partial \bar{\psi}}{\partial t} = \frac{i}{\hbar} \psi \mathcal{H}^+ \]

\[\frac{dP}{dt} = \frac{i}{\hbar} \int (\psi^* \mathcal{H}^+ \psi - \psi \mathcal{H}^+ \bar{\psi}) \quad \text{since} \quad \mathcal{H} = \mathcal{H}^+ \]

\[= \frac{i}{\hbar} \int \psi^*(\mathcal{H}-\mathcal{H})\psi = 0 \]

thus we see the key to conserving probability is to have a dynamical equation that is first order in time.

\[\text{Dirac} \]

\[i\hbar \frac{\partial}{\partial t} \psi = -i\hbar \bar{\alpha} \cdot \bar{\nabla} + \beta mc \psi \]

\[\mathcal{H}_\text{Dirac} = -i\hbar \bar{\alpha} \cdot \bar{\nabla} + \beta mc \]

This equation has the advantage that (1) it is first order in time and (2) has a symmetric treatment of space and time derivatives.
If we apply it twice we should get a KG like equation

\[
(i\hbar \frac{\partial}{\partial t})(i\hbar \frac{\partial}{\partial t}) \psi =
\]

\[
(i\hbar \frac{\partial}{\partial t})(-i\hbar \vec{\alpha} \cdot \vec{\nabla} \psi + \beta mc \psi) =
\]

\[
\int (-i\hbar \vec{\alpha} \cdot \vec{\nabla})(-i\hbar \vec{\alpha} \cdot \vec{\nabla} + \beta mc) + \beta mc (-i\vec{\alpha} \cdot \vec{\nabla} + \beta mc) \psi = 0
\]

\[
\int -\hbar^2 \frac{1}{2} \left( \alpha_i \alpha_j + \alpha_j \alpha_i \right) \delta_{ij} \psi \quad - i\hbar mc \left( \alpha_i \beta \delta_{ij} + \beta \alpha_j \psi \right) + \beta^2 mc^2 \psi \Rightarrow
\]

\[
-\hbar^2 \nabla^2 + mc^2 \psi = -E^2
\]

to recover the KG equation we require

\[
\{ \alpha_i, \alpha_j \} = 2 \delta_{ij}
\]

\[
\{ \alpha_i, \beta \} = 0
\]

\[
\beta^2 = 1
\]

These quantities cannot be ordinary numbers. Dirac chose them to be matrices

\[
Tr(\alpha_i) = Tr(\alpha_i \beta^2) = Tr(\beta \alpha_i \beta) = Tr(-\alpha_i \beta^2) = -Tr(\alpha_i)
\]

\[
Tr(\alpha_i) = 0
\]

\[
\alpha_i^2 - 1 = 0
\]
taken together these equations imply $\lambda$ has eigenvalues $\pm 1$
and there are equal number of $+1$ and $-1$ eigenstates.

& $\Phi$ even dimensional matrices $\lambda$

$\lambda$ Hermitian

For $N=2$ there are only 3 anti commutating
Hermitian matrices $\sigma_i$.

The smallest allowed matrix has
$N=4$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the consistent choice chosen
by Dirac.

$$i\hbar \frac{\partial \psi_a}{\partial t} = (\bar{\alpha}_i \beta + \beta \alpha_i c)_{ab} \psi_b$$

Because this is a Hermitian Hamiltonian
with an equation that is first
order in time =0 probabilities are
conserved.
we also need to consider the transformation properties

\[ i \hbar \frac{\partial \psi}{\partial t} = (\bar{\alpha} \cdot \beta + \beta \rho \gamma^5) \psi \]

left multiply by \( \beta \) to get

\[ (i \hbar \beta \frac{\partial}{\partial t} + i \hbar \beta \bar{\alpha} \cdot \bar{\beta} - \rho \gamma^5) \psi = 0 \]

\[ (i \hbar \gamma^\mu \gamma_\mu - \rho \gamma^5) \psi = 0 \]

\( \gamma^\mu = (\beta, \beta \bar{\alpha}) \)

Note that

\[ \gamma^0 \gamma^0 = \beta^2 = I \]
\[ \gamma^i \gamma^i = \beta \alpha^i \beta \alpha^i = -\beta \beta^i \alpha^i = -I \]
\[ \gamma^0 \gamma^i = \beta \beta \alpha^i = -\beta \alpha^i \beta = -\gamma^i \gamma^0 \]
\[ \gamma^i \gamma^j = \beta \alpha^i \beta \alpha^j = -\beta \beta \alpha^i \alpha^j = +\beta \beta \alpha^i \alpha^j \]
\[ = -\beta \alpha^j \alpha^i = -\gamma^j \gamma^i \quad (i \neq i) \]

\[ \sum \gamma^\mu \gamma^\nu \gamma^5 = -2 \eta^{\mu \nu} \]

we assume that under Lorentz transformations

\[ \psi_a(x) \rightarrow \psi_a'(x') = S(x) \psi_a(x) \]

where \( S(x) \) is a 4x4 matrix representation
of the Lorentz group

\[(i \hbar \gamma^\mu \partial_\mu - m c) \psi'(x') = \]

\[(i \hbar \gamma^\mu \partial_\mu - m c) s(u) \psi'(\Lambda^{-1} x') \]

Note \( \partial_\mu \psi'(\Lambda^{-1} x') = \frac{2}{\partial x_\mu} \psi'(\Lambda^{-1} x') = \frac{2}{\partial x_\mu} \Lambda^{-\alpha}_\mu \psi(x) \)

\[= \Lambda^{-\alpha}_\mu \frac{2}{\partial x_\alpha} \psi(x) \]

\[(i \hbar \gamma^\mu \Lambda^{-\alpha}_\mu \partial_\alpha - m c) s(u) \psi(x) = \]

\[s(u)s(u^{-1}) (i \hbar \gamma^\mu \Lambda^{-\alpha}_\mu \partial_\alpha - m c) s(u) \psi = \]

\[i \hbar (s(u) \gamma^\mu \Lambda^{-\alpha}_\mu s(u^{-1}) \partial_\alpha - m c) \psi(x) = \]

\[\gamma^\alpha = s(u^{-1}) \gamma^\mu s(u) \Lambda^{-\alpha}_\mu \]

\[= s(u^{-1}) (\Lambda^{-1})^\alpha_\mu \gamma^\mu s(u) \]

\[s(u) \gamma^\mu s(u) = \Lambda^\alpha_\mu \gamma^\alpha \]

If we can find \( s(u) \) satisfying this then the Dirac equation has the same form in all coordinate systems.