Relativity and quantum mechanics

\[ x^\mu = (c t, x, y, z) = \text{space-time coordinates of an event} \]

\[ \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ x_\mu \equiv \eta_{\mu\nu} x^\nu = (-ct, x, y, z) \]

\[ A^\mu B_\mu = \sum_0^3 A^\mu B_\mu \quad \text{summation convention} \]

\[ c^2 \Delta_{AB} = \left( x^0_A - x^0_B \right)^2 - \left( \vec{x}_A - \vec{x}_B \right)^2 \]

\[ = c^2 \left( t_A - t_B \right)^2 - \left( \vec{x}_A - \vec{x}_B \right)^2 \]

Einstein preserved under changes in inertial coordinate systems

\[ f^\mu(x) \]

\[ \eta_{\mu\nu} (f^\mu(x) - f^\mu(y)) (f^\nu(x) - f^\nu(y)) = \]

\[ \eta_{\mu\nu} (x^\mu - y^\mu) (x^\nu - y^\nu) \]

Differentiate with respect to \( x^\nu \)

set \( x \) to \( 0 \)

\[ \eta_{\mu\nu} \frac{\partial f^\mu}{\partial x^\nu(0)} (f^\nu(0) - f^\nu(y)) + \]

\[ \eta_{\mu\nu} (f^\mu(0) - f^\mu(x)) \frac{\partial f^\nu}{\partial x^\mu(0)} = \]

\[ \eta_{\mu\nu} s^\mu \left( -y^\nu \right) + \eta_{\mu\nu} (-y^\mu) s^\nu \]
Using $\nabla_{\nu} = \nabla_{\mu}$

$$\nabla_{\nu} \frac{\partial f}{\partial x_{\lambda}} (u) \left( f^{\nu}(u) - f^{\nu}(v) \right) = -\nabla_{\nu} f^{\nu}$$

differentiate with respect to $v_{\lambda}$

$$\nabla_{\nu} \frac{\partial f}{\partial x_{\lambda}} (u) \frac{\partial f}{\partial x_{\lambda}} (v) = \nabla_{\nu} f^{\nu} = \nabla_{\nu}$$

If I term of these quantities as matrices

$$\Lambda^{\mu}_{\nu} = \frac{\partial f^{\mu}}{\partial x_{\nu}} \quad A^{\mu} = f^{\mu}(u)$$

$$\Lambda^{T} \eta \Lambda = \eta$$

This shows that

$$\eta \Lambda^{T} \eta \Lambda = \eta^{2} = I \quad \Lambda^{-1} \eta (\Lambda \eta) = I$$

or

$$\eta \Lambda^{T} \eta = \Lambda^{-1}$$

The equation of the

$$\Lambda^{T} \eta (A - f(u)) = -\eta Y$$

multiply by $\Lambda m$

$$\left( A - f(u) \right) = -\Lambda m \eta Y = -\eta Y$$

$$f(u) = \Lambda \eta + q$$

$$f^{\mu}(u) = \Lambda^{\mu}_{\nu} Y^{\nu} + A^{\mu}$$
\[ y^u \rightarrow y'^u = \Lambda^u_v y^v \]

is called a Lorentz transformation.

\[ y^u \rightarrow y'^u = \Lambda^u_v y^v + a^u \]

is called a Poincaré transformation.

\[ \lambda^\top \eta \lambda = \eta \Rightarrow \]
\[ \det \lambda^\top \det \eta \det \lambda = \det \eta \]
\[ \Rightarrow \]
\[ \det \lambda^\top = \det \lambda = \eta \]
\[ (\det \lambda)^2 = 1 \]
\[ \Rightarrow \]
\[ \det \lambda = \pm 1 \]
\[ \lambda^0, \eta_{00}, \lambda^0, + \lambda^i, \eta_{ii}, \lambda^i = \eta_{00} \]
\[ - (\lambda^0)^2 + \sum (\lambda^i)^2 = -1 \]
\[ (\lambda^0)^2 = 1 + \sum (\lambda^i)^2 \geq 1 \]
\[ \mid \lambda^0 \geq 1 \text{ or } \lambda^0 \leq -1 \]

Lorentz group has 4 disconnected components — can continuously get from one component to the next.
$\Lambda^0 \geq 1 \quad \det \Lambda = 1 \quad \text{proper Lorentz group}$

$\Lambda^0 \leq 1 \quad \det \Lambda = 1 \quad \text{includes space + time reflection}$

$\Lambda^0 \geq 1 \quad \det \Lambda = -1 \quad \text{includes space reflection}$

$\Lambda^0 \leq 1 \quad \det \Lambda = -1 \quad \text{includes time reflection}$

Since the weak interaction does not respect space or time reflection, these are not considered as special relativity.

Treated coordinates related by Poincare transformations as inertial requires modifying Newton's laws.

4 vectors transform like $x^u$

$x^u \rightarrow x'^u = \Lambda^{uv} x^v + a^u$

$$\frac{dx^u}{d\tau} = \frac{dx'^u}{d\tau'} = \Lambda^{uv} \frac{dx^v}{d\tau}$$

$$\frac{d^2x^u}{d\tau^2} = \frac{d^2x'^u}{d\tau'^2} = \Lambda^{uv} \frac{d^2x^v}{d\tau^2}$$

$$\frac{dx^u}{d\tau} = v$$ velocity

$$\frac{d^2x^u}{d\tau^2} = a$$ acceleration

$$\frac{dx^u}{d\tau} = p$$ momentum

These are all 4 vectors.
\[
\frac{dx^u}{dy} = \left( c \frac{dt}{ds}, \frac{dx}{ds} \right) = \left( c \frac{dt}{ds}, \frac{dx}{dt} \frac{dt}{ds} \right) = \left( c \sqrt{1 - \frac{v^2}{c^2}} \right) \frac{dt}{ds}.
\]

\[
c^2 \Delta s^3 = c' \Delta t^3 - \Delta x^3,
\]

\[
\frac{\Delta s^3}{\Delta t^3} = \frac{c'}{c} - \left( \frac{\Delta x}{\Delta t} \right)^3 \frac{1}{c^2},
\]

\[
\left( \frac{\Delta t}{\Delta s} \right) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{take the limit } \Delta t \to 0,
\]

\[
\frac{dt}{ds} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \gamma
\]

\[
\frac{dx^u}{ds} = \left( \gamma c, \gamma \vec{v} \right)
\]

We also have

\[
\frac{d^2x^u}{ds^2} = \left( \frac{d^2x}{ds^2}, \frac{d}{ds} \vec{v} + \gamma \frac{d\vec{v}}{dt} \frac{dt}{ds} \right) = \left( \frac{d^2x}{ds^2}, \frac{d}{ds} \vec{v} + \gamma \frac{d\vec{v}}{dt} \right)
\]

\[
\begin{align*}
\frac{d\vec{v}}{dt} &= \frac{1}{2} \gamma^3 \left( -\frac{2v}{c^2} \frac{dv}{ds} \right) = -\gamma^3 \frac{v}{c} \frac{dv}{dt} \frac{dt}{ds} = -\gamma^3 \beta \frac{dv}{dt} \\
\text{where} \quad \beta &= \frac{v}{c}
\end{align*}
\]

\[
\frac{d^2x^u}{ds^2} = \left( -\gamma^3 \beta \frac{dv}{dt} c, -\gamma^3 \beta \frac{dv}{dt} \vec{v} + \gamma^2 \frac{d\vec{v}}{dt} \right)
\]
There are 2 observations:

1. \( \frac{d^2 x^u}{d \tau^2} \) transforms like a Lorentz 4 vector.

2. \( \frac{d^2 x^u}{d \tau^2} \to (0,0,1,dv/d\tau) = (0, dv/d\tau) \)

If we assume that Newton's laws hold in the particle's instantaneous rest frame, then

\[ (0, dv/d\tau) = (0, \vec{v}) \]

or

\[ (0, 0) = (0, dv/d\tau - \vec{v}) \]

If we assume this is a 4 vector, it must be 0 in every frame:

\[ \frac{d^3 x^u}{d \tau^2} = \Lambda^0_0 \varepsilon \Gamma_i \]

where

\[ \frac{d^3 x^u}{d \tau^2} = \Lambda^0_i \frac{dv^i}{d\tau} \]

We can let \( \Lambda = \Lambda(v) \), we have Lorentz transforms \( \Lambda \) that take

\[ \frac{dx^u}{d\tau} \to (0, \vec{v}) \]

\[ \Lambda(v) (\vec{v}) = (\vec{v}c, \vec{v}) \]
This transformation is not unique since

\[
\Lambda(v) R(u) (e_\vec{0}) = \Lambda(v) (e_\vec{0}) = (\gamma u e_v)
\]

\[
\Lambda'_{\vec{u}}
\]

where \( R(u) \) is a rotation that may depend on \( u \).

One choice is called the canonical boost

\[
\Lambda(v) = \begin{pmatrix}
\gamma(v) & \gamma(v) \beta \\
\beta \gamma & \delta_{ij} + \frac{v_i v_j}{c^2} (\gamma - 1)
\end{pmatrix}
\]

\[\gamma_{\vec{u}} = \Lambda_{\vec{u}}(0,e_\vec{0})\]

\[
\frac{d \mathbf{P}^\mu}{ds} = (\gamma(v)\beta u, \vec{E}, \vec{F}, \vec{\gamma}(\gamma - 1) \beta \cdot \vec{F}) = s^\mu
\]

\[\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\]

Relativistic sum of Newton's laws
Relativistic QM

1. Poincare Group = Symmetry Group

2. Probabilities for equivalent measures the same in all inertial coordinate systems

   \[ \mathbf{U}(\mathbf{u}, \mathbf{q}) \]

   \[ \mathbf{U}(\mathbf{u}, \mathbf{a}) \mathbf{U}(\mathbf{u}, \mathbf{q}) = \mathbf{U}(\mathbf{u}, \mathbf{a}, \mathbf{u} \mathbf{q} + \mathbf{a}) \]

   \[ \mathbf{U}(\mathbf{u}, \mathbf{a})^\dagger = \mathbf{U}(\mathbf{u}, \mathbf{a})^{-1} \]

   Analogy with rotations

   \[ \Delta \mathbf{a} \Delta \mathbf{r} = \mathbf{0} \]

   \[ \mathbf{U}(\mathbf{u}) \]

   One parameter subgroups, with nonmetrical generator

   \[ \mathbf{U}(\mathbf{r}) \]

   \[ \mathbf{U}(\mathbf{r}) \mathbf{J} \mathbf{U}^\dagger(\mathbf{r}) = \mathbf{R}^{-1} \mathbf{J} \mathbf{R} \]

   (J is a vector)

   Under rotation

   \[ \mathbf{J}^2 \text{ invariant} \]

   \[ \mathbf{J} \mathbf{J}^\dagger \text{ commuting observables} \]

   \[ \mathbf{J}_z \text{ change } \mathbf{J}_z \text{, leave } \mathbf{J}^2 \text{ unchanged} \]

   Angular mom

   In rest frame
\[ M^{\mu\nu} = \begin{pmatrix} 0 & 0 & K & K \\ -K & 0 & J_2 & J_y \\ -K & -J_2 & 0 & J_x \\ -K & J_2 & J_x & 0 \end{pmatrix} \]

\[ \mathcal{J}^i = \frac{1}{2} \epsilon^{ijk} \Lambda_{\nu}^{(v)} \Lambda_{\mu}^{(j)} M^{\mu\nu}. \]

Mass \( m \), spin \( \frac{1}{2} \) basis

\( 1(m_g) \bar{\delta}_u \rangle \)

Same thing happens here.

* Start with

\( 1(m_g) \bar{\delta}_u \rangle \)

System at rest

\[ U(R) 1(m_g) \bar{\delta}_u \rangle = \]

\( 1(m_g) \bar{\delta}_v \rangle M(R)_{\nu\mu} \)

Inv. due not change under unitary rep. \( R \)

\[ U(R) 1(m_g) \bar{\delta}_u \rangle = \sum 1(m_g) \bar{\delta}_v \rangle D_{\nu\mu}^{ij}(R) \]
Translations of rest states
\[ p^\mu = m \frac{dx^\mu}{d\tau} = (mc \gamma, mv) \]
\[ p^\mu |(mg)_{0\mu} \rangle = (mc, \sigma, \rho, \sigma) |(mg)_{0\mu} \rangle \]
\[ e^{-ip\cdot\sigma} \left( \begin{array}{c} \sigma^m \cr \ell \end{array} \right) |(mg)_{0\mu} \rangle = e^{-ip\cdot\sigma} |(mg)_{0\mu} \rangle \quad (2) \]
\[ \mathcal{U}(\lambda(v)) |(mg)_{0\mu} \rangle = |(mg)_{\tilde{\nu}, \nu} \rangle N(?) \]

*The normalization must be chosen to make the transformation unitary.*

* \[ J_2 = \lambda^{a}(v) \lambda^{b}(u) M^{ab} \]

measure in rest frame if mapped to rest frame by \( \mathcal{U}(u) \)

\[ \langle \Psi | \psi \rangle = 1 = \int \langle \Psi | \psi \rangle \frac{\delta(p^4+m^4)}{\text{invariant}} \]

\[ \int \langle \Psi | \psi \rangle \frac{d^3p}{2\sqrt{p^4-m^4}} \langle P | i \rangle \quad \text{(integrate } p^0) \]

\[ = \langle \Psi | \psi \rangle \frac{1}{\sqrt{2\omega}} \frac{d^3p}{\omega} \frac{1}{\sqrt{2\omega}} \langle P | i \rangle \quad \omega = \sqrt{p^2m} \]

choosing a \( \phi \) function normalization

\[ \mathcal{U}(u) |(mg)_{0\mu} \rangle = |(mg)_{\tilde{\nu}, \nu} \rangle \sqrt{\frac{\omega(p)}{m}} \quad (3) \]

\( \nu \) does not change under \( \mathcal{U}(v) \) from rest frame
using these three unitary transformations of rest state, we can construct $U(\Lambda)$ in any state.

\[
U(\Lambda) | (m_{\phi}) \bar{\sigma} \mu \rangle = \text{group rep property}
\]

\[
U(\Lambda \Lambda') U(\Lambda') | (m_{\phi}) \bar{\sigma} \mu \rangle = \text{group rep property}
\]

\[
U(\Lambda \Lambda') U(\Lambda') | (m_{\phi}) \bar{\sigma} \mu \rangle = \sqrt{m \omega_{\Lambda}} \langle \Psi | \bar{\sigma} \mu \rangle \sqrt{\frac{m}{\omega_{\Lambda}}}
\]

\[
U(\Lambda \Lambda') U(\Lambda') | (m_{\phi}) \bar{\sigma} \mu \rangle = \text{rotation}
\]

\[
U(\Lambda \Lambda') U(\Lambda') | (m_{\phi}) \bar{\sigma} \mu \rangle = D_{\nu \mu}^{\Sigma}(\Lambda \Lambda') \sqrt{\frac{m}{\omega_{\Lambda}}}
\]

we finally get

\[
U(\Lambda) | (m_{\phi}) \bar{\sigma} \mu \rangle = \text{rotation property}
\]

\[
U(\Lambda) | (m_{\phi}) \bar{\sigma} \mu \rangle = D_{\nu \mu}^{\Sigma}(\Lambda \Lambda') \sqrt{\frac{m}{\omega_{\Lambda}}}
\]
This is the correct unitary representation of the Poincaré group in the plane wave basis for a particle of any spin.

This is not the historical approach.

Historical

0. symmetry of solutions $\rightarrow$
   symmetry of equations
   (more than quantum mechanics demand)

* Klein-Gordon-Schrödinger Equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m^2 c^4\right)\psi(x) = 0$$

$$\bar{p} = \frac{\hbar}{i} \vec{V} \quad p^0 = i \frac{\hbar}{\sqrt{2}} \omega$$

$$\left(+ \frac{\hbar^2}{c^2 \omega^2} \frac{\partial^2}{\partial t^2} - \frac{\hbar^2}{c^2 \omega^2} + m^2 c^4\right)\psi(x+t) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + c^2 \nabla^2 + m^2 c^4\right)\psi(x) = 0$$

3. problems

9. $-i p^0 + \vec{p} \cdot \vec{x} \quad -p^0 + c \vec{p} + m^2 c^4 = 0$

$$p^0 = \pm \sqrt{c^2 p^2 + m^2 c^4}$$

Equation has negative energy states
\[ \int \psi^*(x^+ \psi(x^+ \psi^+ \alpha^+ \psi^0 \psi^+ \psi^+ d^3x \]

\[ \frac{d}{dt} \int \psi^* \psi = \int \left( \frac{\partial \psi^* \psi}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} \right) \neq 0 \]

(Normally \( \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H \psi \quad \frac{\partial \psi^*}{\partial t} = \frac{i}{\hbar} \psi^* H^* = \frac{i}{\hbar} \psi H. \]

\[ - \int \psi^* (\frac{i}{\hbar}) (H-H) \psi = 0 \]

But because this equation is not first order in time \( \frac{d\psi}{dt} \neq 0 \).

(Probabilities are not conserved.)

Gave wrong fine structure in hydrogen.

Relativistic Schrödinger Eq.

\[ i \hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4} \psi. \]

Discarded because it did not treat space and time derivative symmetrically.

This is not a quantum requirement—this equation is not.

Dirac equation

* First order in time so probabilities conserve
  \[ i \hbar \frac{\partial \psi}{\partial t} = H \psi \quad H = H^* \]

* \( H \) linear in space derivatives

* \( (\frac{\hbar^2}{2\alpha^2} - \nabla^2) \psi = \text{Klein-Gordon equation} \)
\[
\frac{\partial \Psi}{\partial t} = \left( \bar{\alpha} \cdot \beta (c + \beta m^2) \right) \Psi
\]

where \( \bar{\alpha} = \alpha^+ \quad \beta = \beta^+ \)

\[
\frac{i \hbar}{2} \left( \frac{\partial \Psi}{\partial t} \right) = -\hbar \frac{\partial^2 \Psi}{\partial t^2} = \left( \alpha_r \cdot \beta + \beta m^2 \right) \left( \alpha_r \cdot \beta + \beta m^2 \right) \Psi
\]

\[
= \sum (\alpha_i \alpha_j \beta_i \beta_j \beta^2 + (\alpha_i \beta + \beta \alpha_i) \beta \cdot c + \beta \cdot m^2 \gamma^i \gamma^j \sigma_i \sigma_j) \Psi = 0
\]

\[
= \beta^2 c^2 + m^2 c^4
\]

This requires

\[
\begin{align*}
\alpha_i \alpha_j + \alpha_i \alpha_j &= 2 \delta_{ij} \\
\alpha_i \beta + \beta \alpha_i &= 0 \\
\beta^2 &= 1
\end{align*}
\]

These can be numbers - Dirac assumed Hermitian matrices

Or they have to be linear combination of \( I \) and \( \sigma_i \) - not enough Hermitian matrices that anticommute.

Dimension must be even

\( \alpha^2 = 1, \beta^2 = 1 \) have \(+1\) as eigenvalues

so \( 4 \times 4 \) is the smallest size of Hermitian matrices that satisfy these conditions
\[ \alpha = \begin{pmatrix} \overline{0} \\ \overline{\xi} \\ \overline{\zeta} \end{pmatrix} \quad \beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

are Diracs choice. What is customary now is to multiply the equation by \( \beta \) from the left:

\[
( i \hbar \beta \frac{\partial \psi}{\partial t} + i mc \beta \cdot \frac{\partial \psi}{\partial \mathbf{x}} - mc^2 ) \psi = 0
\]

\[
\gamma^o = \beta \quad \gamma^i = \beta \sigma^i
\]

\[
( i \hbar \gamma^o \frac{\partial \psi}{\partial \mathbf{X}} - mc ) \psi = 0
\]

This is the standard form of the Dirac equation for a free particle.

I will discuss the solution in another lecture but:

1. It leads to conserved probabilities because it is first order in time

2. \( \mathbf{p} - (\mathbf{p} \cdot \mathbf{E}) / \mathbf{c} \) give the correct hydrogen fine structure

3. Correctly predicts magnetic moment of electron

4. still has negative energy states!!
because it is an equation in spin $\frac{1}{2}$, Dirac assumed all negative energy states are filled. Removing an from the Fermi sea give positive charged electron

\[ \times \text{predicted antielectron} \]

\[ \text{Bound state of electrons + protons are bound, - they are not protected by the filled Fermi sea.} \]

The Dirac equation also need to be reinterpreted = classical field equation (like Maxwell's equations) that gets quantized