Lecture 10

Electrostatic Potentials

$$V(\bar{r}) = k \sum_{n=1}^{N} q_n \frac{1}{|\bar{r}-\bar{r}_n|}$$

The electric field at $\bar{r}$ is

$$\bar{E}(r) = -\bar{\nabla} V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right)$$

This can be generalized to treat continuous charge distributions by replacing

$$\sum_{n=1}^{N} q_n \to \int_V \rho(\bar{r}')\,dr' \qquad V = \text{volume}$$

$$\int_S \sigma(\bar{r}')\,dA' \qquad S = \text{area}$$

$$\int_\gamma \lambda(\bar{r}')\,ds \qquad \gamma = \text{line charge}$$

example: potential due to a circular line charge

$$\lambda = \frac{Q}{2\pi R} \qquad dQ = \lambda R\, d\phi$$

$$V(\bar{r}) = \lambda k R \int_0^{2\pi} \frac{d\phi}{|\bar{r} - \bar{r}'|} =$$

$$= \lambda k R \int_0^{2\pi} \frac{d\phi}{\left((x - R\cos\phi)^2 + (y - R\sin\phi') + z^2\right)^{1/2}}$$

If we set $x = y = 0$ then this becomes

$$V(0\,0\,z) = \lambda k R \int_0^{2\pi} \frac{d\phi}{(R^2+z^2)^{1/2}} = \frac{2\pi\lambda k R}{(R^2+z^2)^{1/2}} = \frac{Qk}{(z^2+R^2)^{1/2}}$$

from this we derive

$$E_z(0\,0\,z) = -\frac{\partial}{\partial z} V(0\,0\,z) = Qk\left(-\frac{\partial}{\partial z}\right)(z^2+R^2)^{-1/2}$$

$$= Qk(-1)\left(-\tfrac{1}{2}\right)(z^2+R^2)^{-3/2}(2z)$$

$$= \frac{kQz}{(z^2+R^2)^{3/2}}$$

which we previously derived from coulomb's law

example 2 potential due to a dipole

$-q$ d q z

$$V = \frac{kq}{|\bar{r} - d/2\,\hat{z}|} - \frac{kq}{|\bar{r} + \frac{d}{2}\hat{z}|} =$$

$$kq\left(\frac{1}{\sqrt{r^2 - d\bar{r}\cdot\hat{z} + d^2/4}} - \frac{1}{\sqrt{r^2 + d\bar{r}\cdot\hat{z} + \frac{d^2}{4}}}\right)$$

$$\frac{kq}{r}\left(\frac{1}{\sqrt{1 - \frac{d\bar{r}}{r^2}\cdot\hat{z} + d^2/4r^2}} - \frac{1}{\sqrt{1 + \frac{d\bar{r}}{r^2}\cdot\hat{z} + d^2/4r^2}}\right)$$

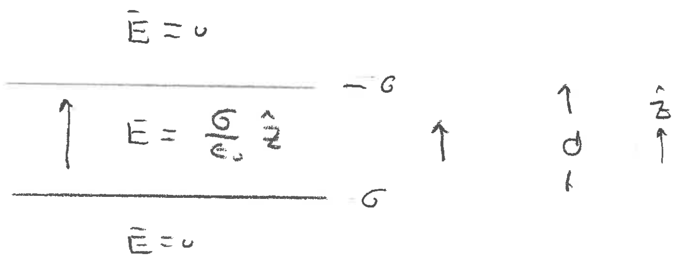
$$\frac{kq}{r}\left(\frac{1}{2}\frac{\bar{r}\cdot\hat{z}\,d}{r^2} + \frac{1}{2}\frac{\bar{r}\cdot\hat{z}\,d}{r^2} + \cdots\right) = \frac{k\,\bar{p}\cdot\bar{r}}{r^3}$$

$$V_{dipole} = k \frac{\bar{p} \cdot \bar{r}}{r^3} = k \frac{p r \cos\theta}{r^3}$$

$$\bar{E}_{dipole} = -\bar{\nabla}\left(k \frac{\bar{p} \cdot \bar{r}}{r^3}\right)$$

where $\cos\theta = \hat{r} \cdot \hat{z}$

infinite parallel plates



In this case the field between the plates is constant in the z direction

The work done in moving a positively charged particle from the upper plate to the lower plate

$$W = (-q \bar{E}) \cdot (-\hat{z} d) = E q d$$

$$V = \frac{W}{q} = E d = \frac{\sigma}{\epsilon_0} d$$

This relates the field, electric potential difference between the plates, the separation between the plates and the charge density

In general a constant potential surface is called an equipotential surface

Since $\bar{E} = -\bar{\nabla} V$

if $\hat{\omega}$ is tangent to a surface of constant potential

$$\hat{\omega} \cdot \bar{\nabla} V = \frac{\partial V}{\partial \omega} = 0$$

because V does not change as ω increases, then

$$\hat{\omega} \cdot \bar{E} = -\hat{\omega} \cdot \bar{\nabla} V = 0$$

$\therefore$ Electric field direction is always $\perp$ to equipotential surfaces

since the field has no components tangent to an equipotential surface

$\boxed{\bar{E} \cdot d\bar{r} = 0 \text{ if } d\bar{r} \text{ is along the equipotential surface}}$

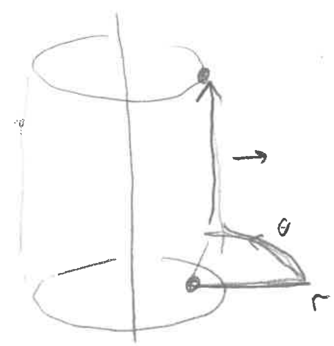
We can use these concepts to calculate the potential if we know the electric field.

main steps

① since the potential is $V - V_0 = -\int_{\bar{r}_0}^{\bar{r}} \bar{E} \cdot d\bar{r}$ in independent of path choose the simplest path

② since $\bar{E} \cdot d\bar{r}$ vanishes for paths along equipotential surfaces, use them to compute the integral

potential due to a infinite charged wire



Gauss' Law

$$2\pi r h \bar{E} = \frac{\lambda h}{\epsilon_0}$$

$$\bar{E} = \frac{\lambda}{2\pi \epsilon_0} \cdot \frac{1}{r}$$

consider the path from

$(r_1\ 0\ 0)$ to (x, y, z)

$r_1 \rightarrow r$ along x direction

$(r\ 0) \rightarrow (r\cos\theta, r\sin\theta, 0)$ in θ direction

$(r\cos\theta\ r\sin\theta, 0) \rightarrow (r\cos\theta\ r\sin\theta, z)$ in z direction note no work is done for the last 2 steps because the paths are on an equipotential surface

$$V(r,0,0) = -\int_{r_1}^{r} E_x \, dx$$

$$= -\int_{r_1}^{r} \frac{\lambda}{2\pi\epsilon_0} \frac{dx}{x}$$

$$= \frac{\lambda}{2\pi\epsilon_0} (\ln r_1 - \ln r)$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_1}{r}\right)$$

$$V(r\cos\theta, r\sin\theta, 0) = V(r,0,0) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_1}{r}\right)$$

$$V(r\cos\theta, r\sin\theta, z) = V(r\cos\theta, r\sin\theta, 0)$$

$$\therefore \quad V(x,y,z) = -\frac{\lambda}{2\pi\epsilon_0} \ln\sqrt{x^2+y^2} + \text{constant}$$

check

$$-\frac{\partial V}{\partial x} = -\left(-\frac{\lambda}{2\pi\epsilon_0}\right)\frac{1}{2}\frac{2x}{x^2+y^2} = \frac{\lambda x}{2\pi\epsilon_0 (x^2+y^2)}$$

$$-\frac{\partial V}{\partial y} = \frac{\lambda y}{2\pi\epsilon_0 (x^2+y^2)}$$

For an infinitely charged plane

Gauss law

$$\bar{E} = \frac{\sigma}{2\epsilon_0} \begin{cases} \hat{z} & z > 0 \\ -\hat{z} & z < 0 \end{cases}$$

start at the origin

$$V(x y 0) - V(000) = -\int_0^x E_x(x' 0, 0)\, dx' - \int_0^y E(x, y' 0)\, dy$$

$= 0$ since the field has no x, y component

$$V(x y z) - V(x y 0) = \begin{cases} -\int_0^z \underbrace{E_z(x y z')}_{\frac{\sigma}{2\epsilon}} dz' = \frac{-\sigma z}{2\epsilon} \\ -\int_0^{-z} \underbrace{E_z(x y z')}_{-\frac{\sigma}{2\epsilon}} (-dz') = \frac{\sigma}{2}\frac{z}{\epsilon} \end{cases}$$

$$V(x y z) = -\frac{\sigma}{2\epsilon_0}|z|$$

$$-\frac{\partial V}{\partial x} = -\frac{\partial V}{\partial y} = 0$$

$$-\frac{\partial V}{\partial z} = \begin{cases} \frac{\sigma}{2\epsilon_0} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} & z < 0 \end{cases}$$

$$\bar{E} = \left(0, 0, \pm \frac{\sigma}{2\epsilon_0}\right)$$

which recovers the field

conductors

* since the electric field inside of a conductor vanishes,

$$\int \bar{E} \cdot d\bar{s} = 0$$

along any path in the conductor

This means the potential difference between any two point in the conductor. This means that the conductor is an <u>equipotential surface</u>.

If we consider the field on the surface

$$\bar{E} = \bar{E}_{parallel} + \bar{E}_{\perp}$$

there cant be a field parallel to the surface because the potential is the same at all points on the surface.

The means that the electric field at the surface of a conductor is <u>normal</u> to the surfa

The means that the electric field on the surface of a conductor is

$$\bar{E} = (\hat{n} \cdot (-\nabla V))\, \hat{n} = \frac{\sigma}{\epsilon_0}$$

in this case since $\bar{E}$ is $\perp$ to the surface

$$-\bar{\nabla} V$$

is in the normal direction, this means

$$\boxed{\sigma(r) = \epsilon_0 (-\hat{n} \cdot \bar{\nabla} V)}$$

This relates the derivitive of potential in the direction normal to the surface to the charge density on the surface.

Potential energy of a charge distribution.

$$V(\bar{r}) = \sum k q_n \frac{1}{|\bar{r} - \bar{r}_n|}$$

$$qV(\bar{r})$$

gives the potential energy of a particle of charge q in this distribution. A related question is what is the potential energy of the entire charge distribution

We assume 0 potential energy corresponds to the situation that all of the charges are infinitely separated

(1) Since there are no other charges present the work needed to bring q_1 to $\bar{r}_1$ is 0.

(2) To bring the second charge it is only necessary to do work against the first charge

$$W_{(2)} = \frac{kq_1q_2}{|\bar{r}_1-\bar{r}_2|}$$

(3) For the third charge we need to work againts the forces due to the first 2 charges

$$W_{(3)} = \frac{kq_3q_1}{|\bar{r}_3-\bar{r}_1|} + \frac{kq_3q_2}{|r_3-r_2|}$$

adding $W_{(2)}$ gives

$$W = \frac{kq_1q_2}{|r_1-r_2|} + \frac{kq_1q_3}{|r_1-r_3|} + \frac{kq_2q_3}{|r_2-r_3|}$$

Following this derivation - the work needed to assemble N charges $q_1 \cdots q_N$ $\bar{r}_1 \cdots \bar{r}_N$

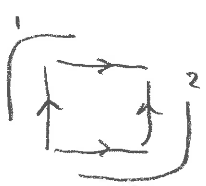
$$\boxed{W = k \sum_{i<j} \frac{q_i q_j}{|r_i - r_j|} = \frac{1}{2} k \sum_{i \neq j} \frac{q_i q_j}{|\bar{r}_i - \bar{r}_j|}}$$

If we have a continuous charge distribution is is replaced by

$$\boxed{W = \frac{1}{2} k \int dV\, dV' \frac{\rho(\bar{r})\rho(\bar{r}')}{|\bar{r} - \bar{r}'|}}$$

We can tell if an electric field $\bar{E}(r)$ comes from charges. We know that the work done against the field is independent of path.

If we calculate the work done against 2 paths and they do not agree, the field is not due to a charge distribution



$$W_1 = q\int_0^L E_y(0\,y)\,dy + q\int_0^L \bar{E}_x(x, L)\,dx$$

$$W_2 = q\int_0^L E_x(x, 0)\,dx + q\int_0^L E_y(L\,y)\,dy$$

We must have

$$W_1 = W_2$$

One check is

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y}$$

since

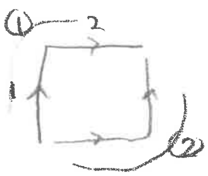
$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} \Rightarrow \boxed{\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}}$$

consider

$$E_x = a y \qquad E_y = -a x$$

$$\frac{\partial E_x}{\partial y} = a \qquad \frac{\partial E_y}{\partial x} = -a$$

$$\frac{\partial E_x}{\partial y} \neq \frac{\partial E_y}{\partial x}$$



$$W_1 = \int_0^L dy\, (-a\,0) + \int_0^L aL\, dx = aL^2$$

$$W_2 = \int_0^L dx\, (a \cdot 0) + \int_0^L (-aL)\, dy = -aL^2$$

In this case the work is path dependent. For 3 dimensions

$$\left.\begin{aligned} \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} &= 0 \\ \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} &= 0 \\ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} &= 0 \end{aligned}\right\}$$ are all needed for the field to be derivable from a potential

* these conditions get modified if the field also depends on time

$$V(x,y,z) = a(x^2+y^2) - bxyz$$

$$E_x = -\frac{\partial V}{\partial x} = -2ax + byz$$

$$E_y = -\frac{\partial V}{\partial y} = -2ay + bxz$$

$$E_z = -\frac{\partial V}{\partial z} = bxy$$

components of the electric field

reconstruct the potential

$$E_x(000) = E_y(000) = E_z(000)$$

$$V(xyz) = V(xyz) - V(000) =$$

$-\int_{000}^{xyz} \bar{E} \cdot d\bar{r}$ along any path

$$-\int_0^x E_x(x,00)\,dx - \int_0^y E_y(xy0)\,dy - \int_0^z E_z(xyz)\,dz$$

$$-\int_0^x (-2ax)\,dx - \int_0^y (-2ay)\,dy - \int_0^z bxy\,dz$$

$$ax^2 + ay^2 - bxyz$$

In this case we chose the simplest path -

example:

#3 lightning 1×10^9 V

$q = 30$ coulombs

$\Delta U = qV = 30C \; 10^9 \frac{J}{C} = 3 \times 10^{10}$ J

use to accelerate a 1000 kg car from rest

$\frac{1}{2} m v^2 = E = 3 \times 10^{10}$

$v^2 = 6 \frac{10^{10}}{10^3} = 6 \times 10^7$

$v = \sqrt{6 \times 10^7} = 7.74 \times 10^3$

#16 potential at center $V(\infty) = 0$

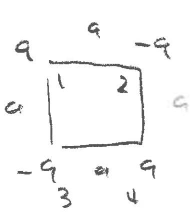
$+2q$ $\quad +4q_2$ $\quad a$ $\quad -3q_1$

a

$-q_1$ $\quad +4q_2$ $\quad +2q$

* $V = k \frac{1}{\sqrt{2}a} \underbrace{(2q_1 - q_1 - 3q_1 + 2q_1)}_{0} + k \frac{1}{a}(4+4) q_2 = \frac{8k}{a} q_2$

#43



$V_{12} + V_{13} + V_{14} + V_{23} + V_{24} + V_{34}$

$-k\frac{q^2}{a} - k\frac{q^2}{a} + k\frac{q^2}{\sqrt{2}a} + k\frac{q^2}{\sqrt{2}a} - k\frac{q^2}{a} - k\frac{q^2}{a}$

$= \frac{kq^2}{a}\left(-4 + \frac{2}{\sqrt{2}}\right) = -\frac{kq^2}{a}\left(4 - \sqrt{2}\right)$