Note Corrections

Lecture 11

Last time

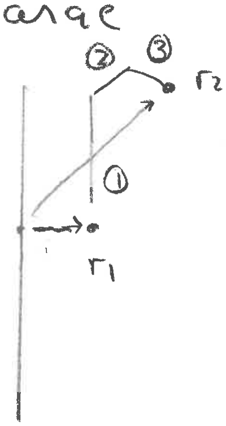
① $\int \bar{E} \cdot ds$ independent of path

② $V(r_2) - V(r_1) = -\int_{\bar{r}_1}^{\bar{r}_2} \bar{E} \cdot ds$

③ $E \perp$ to equipotential surfaces

④ conductors - equipotential, $\bar{E} = 0$

line charge



① $z: (0\,0\,z_1) \to (0\,0\,z_2)$

② $x: (r_1\,0\,z_2) \to (r_2\,0\,z_2)$

③ $\theta: (r\,0\,z_2) \to (r\cos\theta\;\; r\cos\phi,\, z_2) =$

$(x_2\, y_2\, z_2)$

so $V(x_2 y_2 z_2) - V(0\,0\,z_1) =$

$V(x_2 y_2 z_2) - V(r\,0\,z_1) +$

$V(r\,0\,z_1) - V(0\,0\,z_2) +$

$V(0\,0\,z_2) - V(0\,0\,z_1) =$

$0 + \left(-\int_{r_1}^{r_2} E_x(x\,0\,z_z)\,dx\right) + 0$

$(2\pi r h E = h\lambda/\epsilon_0)$ $\quad E = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}$

$\therefore V(x_2 y_2 z_2) - V(0\,0\,z_1) = -\frac{\lambda}{2\pi\epsilon_0}\int_{r_1}^{r_2}\frac{dx}{x} =$

$-\frac{\lambda}{2\pi\epsilon_0}(\ln r_2 - \ln r_1) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_1}{r_2}\right)$

In this way we calculate

$$\boxed{\begin{aligned} &V(\bar{r}_2) = V(0\,0\,z_1) + \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_1}{r_2}\right) \\ &r = \sqrt{x^2+y^2} \end{aligned}}$$

where $V(0\,0\,z_1)$ and r_1 are irrelevant constants. We can choose

$$\boxed{\begin{aligned} V(\bar{r}) &= -\frac{\lambda}{2\pi\epsilon_0}\ln r \\ &= -\frac{\lambda}{2\pi\epsilon_0}\ln\sqrt{x^2+y^2} \\ V(r) &= -\frac{\lambda}{4\pi\epsilon_0}\ln(x^2+y^2) \end{aligned}}$$

This gives the correct field.

$$\bar{E}_x = -\frac{\partial V}{\partial x} = \frac{\lambda}{4\pi\epsilon_0}\frac{2x}{(x^2+y^2)} = \frac{\lambda x}{2\pi\epsilon_0(x^2+r^2)}$$

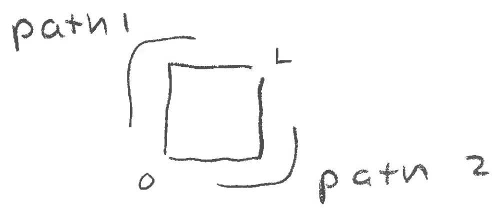
$$\bar{E}_y = \frac{\lambda y}{2\pi\epsilon_0(x^2+r^2)}$$

$$\bar{E}_z = 0$$

The path independence of $\int_A^B \bar{E} \cdot d\bar{r}$ means that the electric field is not arbitrary

example consider

$$\bar{E} = (E_x\ E_y\ E_z) = (ay, -ax, z)$$



$$(1) \quad V(LL) - V(00) = \int_0^L E_y(0, y')\, dy + \int_0^L E_x(x', L)\, dx'$$
$$= 0 + (aL^2)$$

$$(2) \quad V(LL) - V(00) = \int_0^L E_x(x', 0)\, dx' + \int_0^L E_y(L, y')\, dy'$$
$$+ \ 0 + (-aL^2)$$

in this case both paths give different answers – why

$$E_x = -\frac{\partial V}{\partial x} \quad E_y = -\frac{\partial V}{\partial y} \Rightarrow$$

$$\frac{\partial E_x}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial E_y}{\partial x}$$

$$\Rightarrow \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$$

similarly $\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = 0 \qquad \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0$

This looks like a cross product

$$\bar{\nabla} \times \bar{E} = \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) = 0$$

this is no longer true when $\bar{E}$ depends on time

Potential energy of a charge distribution

To calculate this we assume that the potential at ∞ is 0. We compute the total work needed to assemble the charge distribution

① first charge – no work is needed because there is no field

② second charge

$$W_{12} = q_1 q_2 k \frac{1}{|r_1 - r_2|}$$

③ third charge

$$W = W_{31} + W_{32} = k q_1 q_3 \frac{1}{|r_1 - r_3|} + k q_2 q_3 \frac{1}{|r_2 - r_3|}$$

total work

$$W_T = W_{12} + W_{13} + W_{23} = \sum_{i<j} \frac{k q_i q_j}{|r_i - r_j|} \text{ or}$$

$$\frac{1}{2} \sum_{i \neq j} \frac{k q_i q_j}{|r_i - r_j|}$$

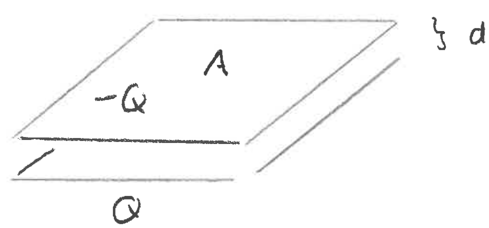
this clearly generalizes

$$E = \frac{1}{2} \sum_{i \neq j} \frac{k q_i q_j}{|r_i - r_j|}$$

Capacitors

capacitors are electronic components that store electric charge and energy.

The concept of capacitance is illustrated by considering a system consisting of 2 parallel conducting plates of area A and separation d.



Assume one plate has charge Q and one has charge -Q

each plate is an equipotential surface, but the upper and lower plate are at different potentials

If we ignore edge effects the charge density in the plate will be approximatey constant $\sigma \simeq \frac{Q}{A}$

If we use gaussian surfaces near the center of the plate where $\sigma \sim Q/A$ the electric field will be

$$EA = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{\epsilon_0}$$

The change in potential between the plates

$$V_- - V_+ = Ed = \frac{\sigma}{\epsilon_0} d = \frac{\sigma A d}{\epsilon_0 A} = \frac{Qd}{\epsilon_0 A}$$

This has the form

$$\Delta V = \text{change in potential}$$

$$= \cancel{C}Q \quad \frac{1}{C} Q$$

where $1/C = \frac{d}{\epsilon_0 A}$ $\quad C = \frac{\epsilon_0 A}{d}$

This shows that the charge one one plate is proportional to the potential difference between the plates. It is conventional to use V for ΔV which gives

$$\boxed{V = \cancel{C}Q} \qquad Q = CV$$

The constant of proportionality C is called the <u>capacitance</u> of the device

for parallel plates $\cancel{C = \frac{d}{\epsilon_0 A}}$ $C = \frac{\epsilon_0 A}{d}$

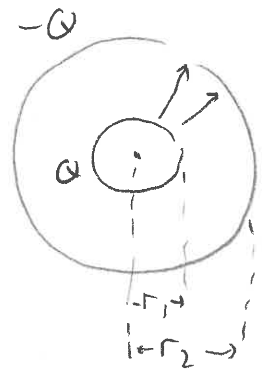
units of capacitance

$$\boxed{1 \text{ Farad} = \frac{1 \text{ }\cancel{\text{Volt}}\text{ Coulom}}{\cancel{\text{Coulomb}}\text{ Volt}} = \frac{1 \text{ Joule}}{(\text{Coulomb})^2}} \quad \frac{(\text{Coulomb})^2}{\text{Joule}}$$

note that the capacitance depends only on geometrical properties of the device - in this case area of 1 plate and the separation of the plates.

We can compute the capacitance for some other simple geometries

Case of concentric conducting spheres of radius r_1 r_2



In this case the electric field between the conductors is radially outward with

$$|\vec{E}(r)| = \frac{kQ}{r^2}$$

The change in potential in going from the outer sphere to the inner sphere is

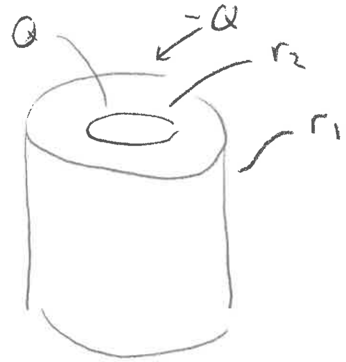
$$-\int_{r_2}^{r_1} \frac{kQ}{r^2}(-dr) = -\frac{kQ}{r}\Big|_{r_1}^{r_2} = kQ\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\Delta V = V = k\left(\frac{1}{r_1} - \frac{1}{r_2}\right)Q = k\left(\frac{r_2 - r_1}{r_1 r_2}\right)Q$$

In this case $C = \frac{1}{k}\,\frac{r_1 r_2}{r_2 - r_1} = 4\pi\epsilon_0 \frac{r_1 r_2}{r_2 - r_1}$

$$\boxed{C = k\left(\frac{r_2 - r_1}{r_1 r_2}\right) = \frac{1}{4\pi\epsilon_0}\left(\cancel{\frac{r_2 - r_1}{r_1 r_2}}\right)}$$

By convention C is chosen to be positive coincentric cylinders



In this case the field is radially outward

The magnitude of the field follow from Gauss law

$$E \times 2\pi r h = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{2\pi r h \epsilon_0}$$

The change in potential going from the outer surface to the inner surface is

$$V_2 - V_1 = -\int_{r_1}^{r_2} \bar{E} \cdot d\bar{r} = -\int_{r_1}^{r_2} \frac{Q}{2\pi r h \epsilon_0} \hat{r} \cdot \hat{r} \, dr$$

$$= -\int_{r_1}^{r_2} \frac{Q}{2\pi h \epsilon_0} \frac{dr}{r}$$

$$= \frac{Q}{2\pi h \epsilon_0} (\ln(r_1) - \ln(r_2))$$

$$= \frac{Q}{2\pi h \epsilon_0} \ln\left(\frac{r_1}{r_2}\right)$$

$$V = \left(\frac{1}{2\pi h \epsilon_0} \ln\left(\frac{r_1}{r_2}\right)\right) Q$$

So in this case the capacitance is

$$\boxed{C = \cancel{\frac{1}{2\pi h \epsilon_0} \ln\left(\frac{r_1}{r_2}\right)}}$$

$$C = 2\pi h \epsilon_0 \frac{1}{\ln(r_1/r_2)}$$

in all three cases the capacitence is geometric

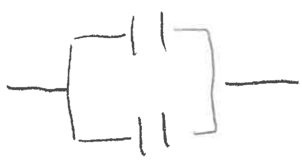
circuits with capacitors

we use the notation —||— for a capacitor

capacitors in series have the form

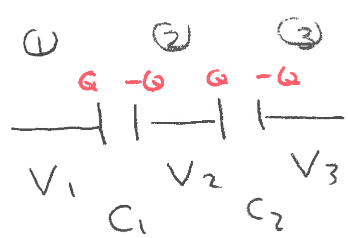
—||—||—

capacitors in parallel have the form



each of these configurations has an equivalent capacitence.

series



$$V_2 - V_1 = \frac{1}{C_1} Q \qquad V_2 - V_1 = \frac{1}{C_1} Q$$

since $Q_1 = -Q_2 = Q_3$

$$V_3 - V_2 = \frac{1}{C_2} Q \qquad V_3 - V_2 = \frac{1}{C_2} Q$$

$$V_3 - V_1 = V_3 - V_2 + V_2 - V_1 = \frac{1}{C_1}Q + \frac{1}{C_2}Q$$

$$V = \frac{1}{(C_1 + C_2)}\,Q$$

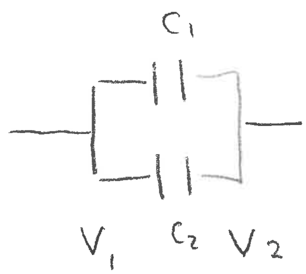
So the effective capacitance is

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{capacitors in series}}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

For the parallel case



$$V_2 - V_1 = \frac{1}{C_1}Q_1 = \frac{1}{C_2}Q_2 = C(Q_1 + Q_2)$$

$$Q_1 = \frac{V}{\cancel{C_1}}C_1 \quad Q_2 = \frac{V}{\cancel{C_2}}C_2 \quad Q_1 + Q_2 = \frac{V}{\cancel{C}}C$$

$$V\left(\cancel{\frac{1}{C_1}}\,C_1 + \cancel{\frac{1}{C_2}}\,C_2\right) = V\left(\cancel{\frac{1}{C}}\,C\right)$$

canceling we get

$$\boxed{C = C_1 + C_2 \quad \cancel{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}} \quad \text{capacitors in parallel}}$$

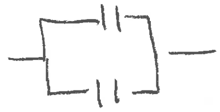
we can write this as

$$\cancel{\frac{1}{C} = \frac{C_2 + C_1}{C_1 C_2} \qquad C = \frac{C_1 C_2}{C_1 + C_2}}$$

$$C = C_1 + C_2$$

example - consider the circuit

find the effective capacitance for this circuit. find the charge on each capacitor

① replace  by C_{23}

$C_2 + C_3 = C_{23}$

$\cancel{\frac{1}{C_{23}} = \frac{1}{C_2} + \frac{1}{C_3}}$ $\qquad C_{23} = \cancel{\frac{C_2 C_3}{C_2 + C_3}}$ $\quad C_2 + C_3$

combine this with the ~~first~~ capacitor =

$$\boxed{C_{TOTAL} = \cancel{C_1 +} \frac{C_2 C_3}{C_2 + C_3}}$$

$\frac{1}{C_{Tot}} = \frac{1}{C_1} + \frac{1}{C_{23}} = \frac{1}{C_1} + \frac{1}{C_2 + C_3}$

$C = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3}$

② since $CV = \cancel{C}Q$

$$\boxed{Q = Q_1 = Q_2 + Q_3 = \frac{VC}{\cancel{C_{TOTAL}}}}$$

$V_{2-m} = \frac{1}{C_2} Q_2 = \frac{1}{C_3} Q_3 \qquad Q_3 = \cancel{\frac{C_2}{C_3}} Q_2 \qquad Q_3 = \frac{C_3}{C_2} Q_2$

$\frac{CV}{\cancel{C_{TOTAL}}} = Q = Q_2 + \frac{C_3}{C_2} Q_2 = \left(\frac{C_2 + C_3}{C_3}\right) Q_2$

$$\boxed{Q_2 = \frac{C_2}{C_2 + C_3} Q \qquad Q_3 = \frac{C_3}{C_2 + C_3} Q}$$

Energy stored in a capacitor

$$dW = dq\,V = dq\,C q$$

$$W = \frac{1}{2}\frac{1}{C}q^2 = \Delta V$$

so the energy stored in a capacitor is proportional to the square of the total charge on the capacitor

Note that it is not possible to store an unlimited charge on a capacitor. Eventually the potential difference will be high enough to pull electrons off of the conductor. The voltage where this happens is called the breakdown voltage