Lecture 12

Capacitors

Corrected formula

\[ Q = CV \]

\( Q = \text{charge on device} \)

\( V = \text{potential difference across device} \)

\( C = \text{capacitance of device} \)

Units

\[ C = \frac{Q}{V} = \frac{\text{Coulomb}^2}{\text{Joule}} = \frac{\text{Coulomb}}{\text{Volt}} = 1 \text{ Farad} \]

The capacitance is a geometric quantity.

Parallel plates \( C = \frac{\varepsilon_0 A}{d} \)

Concentric spheres \( C = 4\pi\varepsilon_0 \frac{r_2}{r_2-r_1} \) \( (r_2>r_1) \)

Concentric cylinders \( C = 2\pi\varepsilon_0 \frac{1}{\ln(r_2/r_1)} \) \( (r_2>r_1) \)

Capacitors can be put in complex circuits, 2 of the building blocks are capacitors in parallel and capacitors in series.
capacitors in series

\[ V_1 \quad V_2 \quad V_3 \]

\[ C_1 \quad C_2 \]

* plates 2 and 3 are at the same potential

\[ Q_1 = C_1 (V_2 - V_1) \]
\[ Q_2 = C_2 (V_3 - V_2) \]

\[ V_3 - V_1 = V = (V_3 - V_2) + (V_2 - V_1) = \frac{Q}{C} \]
\[ = \frac{Q}{C_1} + \frac{Q}{C_2} \]

This gives

\[ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \]
\[ \text{or} \quad C = \frac{C_1 C_2}{C_1 + C_2} \]

capacitors in series

capacitors in parallel

\[ (V_2 - V_1) C_1 = Q_1 \quad (V_2 - V_1) C_2 = Q_2 \]
\[ (V_2 - V_1)(C_1 + C_2) = (Q_1 + Q_2) = Q \]
\[ C_1 + C_2 = C = \frac{Q}{V_2 - V_1} = \frac{Q}{V} \]

for capacitors in parallel

\[ C = C_1 + C_2 \]

These rules can be combined to treat more complex circuits.

Assume that there is a potential \( V \) across this device.

\( 0 \) replace this with

\[ \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2 + C_3} \]

\( C_T = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} \)

\[ Q = V C_T \]

\[ Q = V \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} \]

\[ V_1 = \frac{Q}{C_1} = \frac{1}{C_1} \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} V = \frac{C_2 + C_3}{C_1 + C_2 + C_3} V \]
we can make this more complicated by

\[
V_2 = \frac{Q}{C} = \frac{Q}{C_1+C_2} = \frac{C_1}{C_1+C_2+C_3} V
\]

\[
Q_2 = C_2 V_2 = \frac{C_1 C_2}{C_1+C_2+C_3} V
\]

\[
Q_3 = C_3 V_2 = \frac{C_3 C_1}{C_1+C_2+C_3} V
\]

energy stored in a capacity:

consider a capacity with capacitance \( C \) and charge \( Q \)

\[
Q = CV
\]

\[
V = \frac{1}{C} Q
\]

to move a charge \( dQ \) from the negative plate to the positive plate involve con

\[
dW = V dQ = \frac{1}{C} Q dQ
\]

change in potential energy
The total energy stored in the capacitor is the net work done in raising the potential energy by changing the capacitance.

\[ W = \int_0^Q \frac{1}{C} \, dQ = \frac{Q^2}{2C} \]

The factor of 2 arises because we only have to work against the present charges, not the final total charge.

For a parallel plate capacitor:

\[ C = \epsilon_0 \frac{A}{d} \quad EA = \frac{Q}{\epsilon_0} \quad Q = \epsilon_0 EA \]

\[ W = \frac{\epsilon_0^2 E^2 A^2}{2 (\epsilon_0 A/d)} = \frac{1}{2} \epsilon_0 \frac{E^2}{d} \left( A \cdot d \right) \]

 volume where the field is present.

In this sense, we can think of the field as having an energy/unit volume of

\[ E = \frac{\text{energy}}{\text{volume}} = \frac{1}{2} \epsilon_0 E^2 \]
This is actually a general result.

Consider the case of concentric spheres:

\[ C = 4\pi \varepsilon_0 \frac{r_2 - r_1}{r_2 - r_1} \]

\[ E = \frac{G}{4\pi \varepsilon_0} \frac{1}{r_2} \]

\[ E^2 = \frac{G^2}{(4\pi \varepsilon_0)^2} \frac{1}{r_4} \]

\[ \int \frac{1}{2} \varepsilon_0 E^2 \, dV = \int_{r_1}^{r_2} dr \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \frac{1}{2} \varepsilon_0 \frac{G^2}{(4\pi \varepsilon_0)^2} \frac{1}{r_4} \]

\[ = \frac{4\pi}{2\varepsilon_0} \frac{G^2}{(4\pi)^2} \int_{r_1}^{r_2} \frac{dr}{r^2} \]

\[ = \frac{G^2}{8\pi \varepsilon_0} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) = \frac{G^2}{8\pi \varepsilon_0} \left( \frac{r_1}{r_1^2} - \frac{r_2}{r_2^2} \right) \]

\[ = \frac{Q^2}{2C} \]

You can also check this with the cylinder. This suggests that we can think of the field itself as having energy.
Consider a capacitor - as the charge increases the electric field between the plates increases.

If there are atoms between the plates - the electrons are attracted to the positive plate while the protons are attracted towards the negative plate. If the field is sufficient to ionize the atoms the capacitor can no longer hold additional charge.

\[ Q_{\text{max}} = V_{\text{max}} C \]

\( V_{\text{max}} \) is called the breakdown voltage of the capacitor.
If the material between the plates consists of a bunch of electric dipoles, the presence of the dipoles reduces the strength of the electric field between the plates. Extra layers of charges cancel.

Recall

\[ \frac{Q}{\varepsilon_0} = EA \rightarrow Q = Q_0 - Q_0 \]

\[ \frac{Q - Q_0}{\varepsilon_0} = EA \]

If \( Q_0 \propto d \alpha = d \)

\[ \frac{Q(1-d)}{\varepsilon_0} = EA \]

Which can be fixed by replacing \( \varepsilon_0 \) by \( \varepsilon = \frac{1}{1-d} \varepsilon_0 \).
The dielectric constant \( K \) is defined by \( K = \frac{1}{1 - \epsilon^*} \). Normally \( K > 1 \) if the dielectric reduces the field strength.

\[ Q = \text{change deposited on plate} \]

\[ \frac{1}{K} Q = \text{total change - including polarization change} \]

\[ \oint E \cdot \hat{n} dA = \frac{1}{K \epsilon_0} Q \]

Gauss Law does not change - but it is more convenient to not have to worry about the polarization change.

This leads to a change in Coulomb's law in a dielectric medium.

Consider a spherical surface of radius \( r \) about a point charge

\[ 4\pi r^2 \vec{E} = \oint E \cdot \hat{n} dA = \frac{1}{K \epsilon_0} Q \]

\[ \vec{E} = \frac{Q}{4\pi \epsilon_0 \frac{r^2}{r^2}} \]
If we consider the parallel plate capacitance note Gauss law gives

\[ \mathbf{E} \cdot \hat{n} \, dA = \frac{1}{\kappa \varepsilon} \, Q \rightarrow \]

\[ \mathbf{E} \cdot A = \frac{1}{\kappa \varepsilon} \, Q \]
\[ V = Ed \]

\[ \frac{V}{d} \cdot A = \frac{1}{\kappa \varepsilon} \, Q \]
\[ Q = V \cdot \varepsilon \left( \frac{EA}{d} \right) \]

which shows that the capacitance is increased from

\[ C = \frac{EA}{d} \quad \text{to} \quad C' = \kappa C \]

This is used to make real capacitance—dielectric material is inserted between conductors

\[ \varepsilon = \frac{1}{2} \frac{Q^2}{C} \]

The energy in the capacitance decreases to

\[ \varepsilon' = \frac{1}{2} \frac{Q^2}{C'} = \varepsilon / \kappa \]

(This is for a capacitance with fixed charge.)
Since \( \theta = CV \)

\[
\mathcal{E} = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{C^2 V_i}{C} = \frac{1}{2} V^2 C
\]

If \( C = \kappa C \) keeping \( V \) constant then

\[
\mathcal{E} \rightarrow \mathcal{E}' = \kappa \mathcal{E}
\]

so the energy increases by inserting the dielectric

\[
\text{The capacitance is proportional to } d
\]

\[
C' = C \cdot \frac{L-d}{L}
\]

\[
C' = \kappa C \frac{d}{L}
\]

so as a function of \( d \)

\[
\mathcal{E} = \frac{1}{2} V^2 \left( \kappa C \frac{d}{L} + C \frac{L-d}{L} \right)
\]

\[
d\omega = \mathcal{E}(d+\Delta d) - \mathcal{E}(d) = F \Delta d
\]

\[
\frac{1}{2} V^2 \left( \kappa C - C \right) \Delta d
\]

\[
d\omega = \frac{1}{2} V^2 C (\kappa-1) \frac{L}{L} \Delta d
\]

\[
F = \frac{1}{2} V^2 C (\kappa-1) \frac{L}{L} = \text{force needed to insert dielectric}
\]

\[\text{for fixed potential}\]
Next consider the case of a fixed charge

\[ E = \frac{1}{2} \frac{Q^2}{C} \rightarrow E' = \frac{1}{2} \frac{Q^2}{\kappa C} \]

\[ E_{\text{final}} - E_{\text{initial}} = \frac{1}{2} \frac{Q^2}{C} \left( \frac{1}{\kappa} - 1 \right) \]

\[ = \frac{1}{2} \frac{Q^2}{C} \left( \frac{1 - \kappa'}{\kappa'} \right) \]

\[ = - \frac{1}{2} \frac{Q^2}{C} \left( \frac{\kappa - 1}{\kappa} \right) \]

This is the total work needed to put the dielectric in the plates.

In this case, \( F \cdot L = \) the change in energy so there is an average inward force of

\[ F = \frac{1}{2} \frac{Q^2}{C} \left( \frac{\kappa - 1}{\kappa} \right) \cdot L. \]