Lecture 12

Capacitors

Corrected formula

$$Q = CV$$

Q = charge on device

V = potential difference across device

C = capacitance of device

units

$$C = \frac{Q}{V} = \frac{(\text{Coulomb})^2}{\text{Joule}} = \frac{\text{Coulomb}}{\text{Volt}} = 1 \text{ Farad}$$

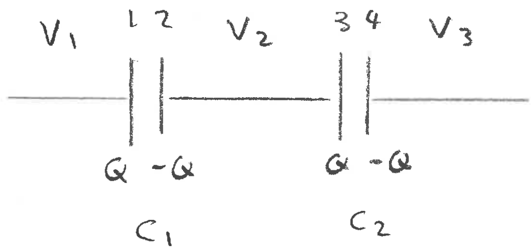
The capacitance is a geometric quantity

parallel plates $\quad C = \dfrac{\epsilon_0 A}{d}$

coincentric spheres $\quad C = 4\pi\epsilon_0 \dfrac{r_1 r_2}{r_2 - r_1} \quad (r_2 > r_1)$

coincentric cylinders $\quad C = 2\pi h \epsilon_0 \dfrac{1}{\ln(r_2/r_1)} \quad (r_2 > r_1)$

Capacitors can be put in complex circuits, 2 of the building blocks are capacitors in parallel and capacitors in series capacitors in series



* plates 2 and 3 are at the same potential

$$Q_1 = C_1 (V_2 - V_1)$$
$$Q_2 = C_2 (V_3 - V_2)$$

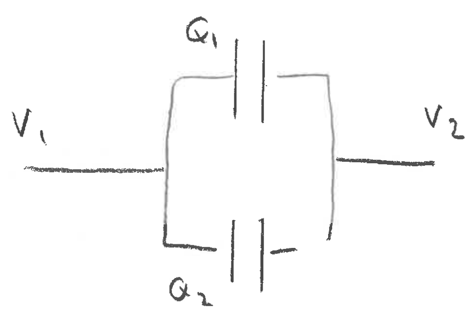
$$V_3 - V_1 = V = (V_3 - V_2) + (V_2 - V_1) = \frac{Q}{C}$$
$$= \frac{Q}{C_1} + \frac{Q}{C_2}$$

This gives

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C = \frac{C_1 C_2}{C_1 + C_2}}$$

capacitors in series

capacitors in parallel



$$(V_2 - V_1) C_1 = Q_1 \qquad (V_2 - V_1) C_2 = Q_2$$

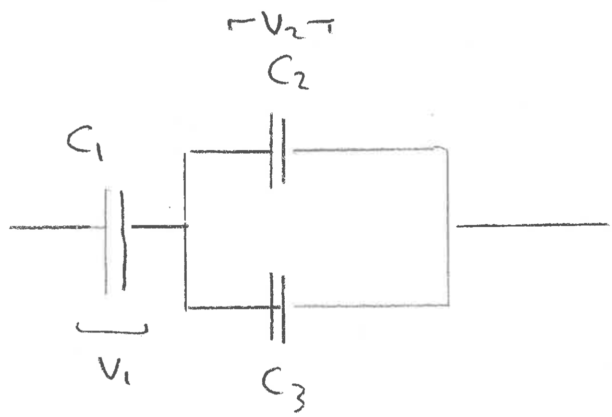
$$(V_2 - V_1)(C_1 + C_2) = (Q_1 + Q) = Q$$

$$C_1 + C_2 = C = \frac{Q}{V_2 - V_1} = \frac{Q}{V}$$

for capacitors in parallel

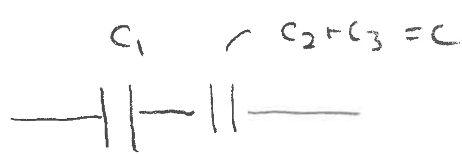
$$\boxed{C = C_1 + C_2}$$

These rules can be combined to treat more complex circuits



Assume that there is a potential V across this device.

① replace this with



$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2 + C_3} = \frac{C_1 + C_2 + C_3}{C_1(C_2 + C_3)}$$

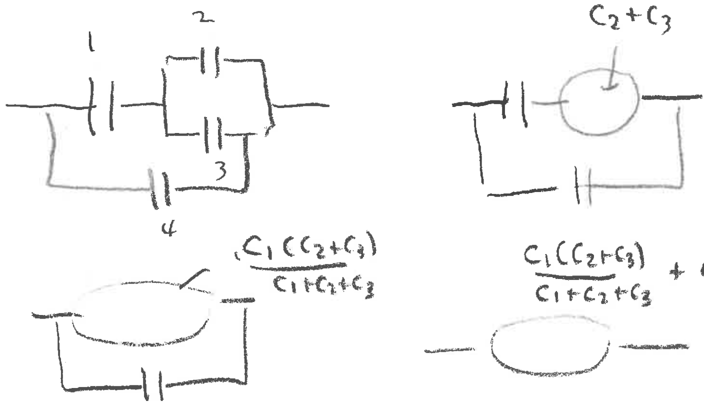
$$Q = V C_T$$

$$\boxed{Q = V \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3}}$$

$$\boxed{V_1 = \frac{Q}{C_1} = \frac{1}{C_1} \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3} V = \frac{C_2 + C_3}{C_1 + C_2 + C_3} V}$$

$$\boxed{\begin{aligned} V_2 &= \frac{Q}{C} = \frac{Q}{C_1+C_2} = \frac{C_1}{C_1+C_2+C_3} V \\ Q_2 &= C_2 V_2 = \frac{C_1 C_2}{C_1+C_2+C_3} V \\ Q_3 &= C_3 V_2 = \frac{C_3 C_1}{C_1+C_2+C_3} V \end{aligned}}$$

we can make this more complicated by



energy stored in a capacitor

consider a capacitor with capacitance C and charge Q

$$Q = CV$$

$$V = \frac{1}{C} Q$$

to move a charge dQ from the negative plate to the positive plate involve work

$$dW = \underbrace{V\,dQ}_{\text{change in potential energy}} = \frac{1}{C} Q\,dQ$$

The total energy stored in the capacitor is the net work done in raising the potential energy = charging the capacitor

$$\boxed{W = \int_0^Q \frac{1}{C} Q\, dQ = \frac{Q^2}{2C}}$$

The factor of 2 arises because we only have to work against the present charges - not the final total charge.

For a parallel plate capacitor

$$C = \frac{\epsilon_o A}{d} \qquad EA = \frac{Q}{\epsilon_o} \qquad Q = \epsilon_o E A$$

$$W = \frac{\epsilon_o^2 E^2 A^2}{2\left(\frac{\epsilon_o A}{d}\right)} = \frac{1}{2} \epsilon_o E^2 \underbrace{(A \cdot d)}$$

Volume where the field is present.

In this sense we can think of the field as having an energy / unit volume of

$$\boxed{\mathcal{E} = \frac{\text{energy}}{\text{volume}} = \frac{1}{2} \epsilon_o E^2}$$

This is actually a general result

consider the case of concentric spheres

$$C = 4\pi\epsilon_0 \frac{r_1 r_2}{r_2 - r_1}$$

$$E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$$

$$E^2 = \frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$

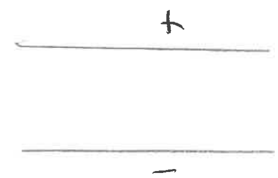
$$\int \frac{1}{2}\epsilon_0 E^2\, dV = \int_{r_1}^{r_2} dr \int_0^{\pi} r\, d\theta \int_0^{2\pi} r\sin\theta\, d\varphi \;\frac{1}{2}\epsilon_0 \frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$

$$= \frac{4\pi}{2\epsilon_0}\frac{Q^2}{(4\pi)^2}\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

$$= \frac{Q^2}{2C}$$

you can also check this with the cylinder. This suggests that we can think of the field itself as having energy



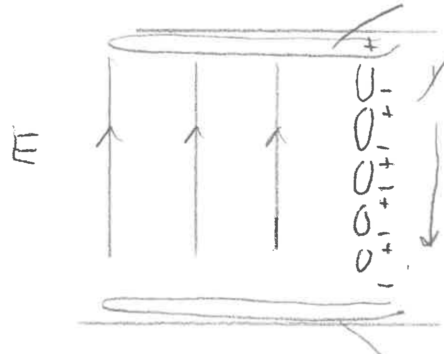
consider a capacitor – as the charge increases the electric field between the plates increases

If there are atoms between the plates – the electrons are attracted to the positive plate while the protons are attracted towards the negative plate. If the field is sufficient to ionize the atoms the capacitor can no longer hold additional charge

$$Q_{MAX} = V_{max} C$$

V_{max} is called the <u>breakdown voltage</u> of the capacitor.

If the material between the plates consists of a bunch to electric dipoles



The presence of the dipoles reduces the strength of the electric field between the plates

recall

$$\frac{Q}{\epsilon_0} = E \cdot A \quad \rightarrow \quad Q \rightarrow Q - Q_D$$

$$\frac{Q - Q_D}{\epsilon_0} = EA$$

If $Q_D \propto \alpha Q \Rightarrow$

$$\frac{Q(1-\alpha)}{\epsilon_0} = EA$$

which can be fixed by replacing

$$\boxed{\epsilon_0 \text{ by } \epsilon = \frac{1}{1-\alpha}\epsilon_0}$$

The dielectric constant κ is defined by $\kappa = \frac{1}{1-\alpha}$. Normally $\kappa > 1$ if the dielectric reduces the field strength

Q = charge deposited on plate

$\frac{1}{\kappa} Q$ = total charge - including polarization charge

$$\oint \bar{E} \cdot \hat{n} \, dA = \frac{1}{\kappa \epsilon_0} Q$$

Gauss Law does not change - but it is more convenient to not have to worry about the polarization charge

This leads to a change in Coulombs law in a dielectric medium

Consider a spherical surface of radius r about a point charge

$$4\pi r^2 E = \oint \bar{E} \cdot \hat{n} \, dA = \frac{1}{\kappa \epsilon_0} Q$$

$$\boxed{\bar{E} = \frac{Q}{4\pi \kappa \epsilon_0} \frac{\hat{r}}{r^2}}$$

If we consider the parallel plate capacitor note gauss law gives

$$\bar{E} \cdot \hat{n}\, dA = \frac{1}{\kappa \epsilon_0} Q \rightarrow$$

$$E A = \frac{1}{\kappa \epsilon_0} Q \qquad V = Ed$$

$$\frac{V}{d} A = \frac{1}{\kappa \epsilon_0} Q$$

$$Q = V \kappa \left(\frac{\epsilon_0 A}{d} \right)$$

which shows that the capacitance is increased from

$$\boxed{C = \frac{\epsilon A}{d} \quad \text{to} \quad C' = \kappa C}$$

This is used to make real capacitors — dielectric material is inserted between conductors

$$\mathcal{E} = \frac{1}{2} \frac{Q^2}{C}$$

the energy in the capacitor decreases to

$$\mathcal{E}' = \frac{1}{2} \frac{Q^2}{C'} = \mathcal{E} / \kappa$$

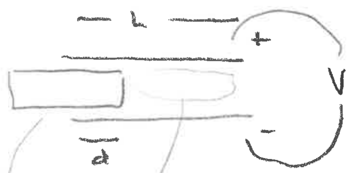
(this is for a capacitor with fixed charge) since $Q = CV$

$$\mathcal{E} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{C^2V^2}{C} = \frac{1}{2}V^2C$$

If $C \to \kappa C$ keeping V constant then

$$\mathcal{E} \to \mathcal{E}' = \kappa\mathcal{E}$$

so the energy increases by inserting the dielectri



the capacitance is proportional to d

$$C' = C\cdot\frac{L-d}{L}$$

$$C' = \kappa C\frac{d}{L}$$

so as a function of d

$$\mathcal{E} = \frac{1}{2}V^2\left(\kappa C\frac{d}{L} + C\frac{L-d}{L}\right)$$

$$dW = \mathcal{E}(d+\Delta d) - \mathcal{E}(d) = F\Delta d$$

$$\frac{1}{2}V^2\left(\frac{\kappa C - C}{L}\right)\Delta d$$

$$dW = \frac{1}{2}V^2C(\kappa-1)\frac{1}{L}\,d\Delta$$

$$F = \frac{1}{2}V^2C(\kappa-1)\frac{1}{L}$$ = force needed to insert dielectri for fixed potential

Next consider the case for fixed charge

$$\mathcal{E} = \frac{1}{2}\frac{Q^2}{C} \rightarrow \mathcal{E}' = \frac{1}{2}\frac{Q^2}{\kappa C}$$

$$\mathcal{E}_{final} - \mathcal{E}_{init} = \frac{1}{2}\frac{Q^2}{C}\left(\frac{1}{\kappa} - 1\right)$$

$$= \frac{1}{2}\frac{Q^2}{C}\left(\frac{1-\kappa}{\kappa}\right)$$

$$= -\frac{1}{2}\frac{Q^2}{C}\left(\frac{\kappa-1}{\kappa}\right)$$

This is the total work needed to put the dielectric in the plates

In this case $F \cdot L$ = the change in energy so there is an average inward force of

$$F = \frac{1}{2}\frac{Q^2}{C}\left(\frac{\kappa-1}{\kappa}\right)\cdot\frac{1}{L}.$$