Lecture 13

So far we have only considered static electric charges — i.e., charges that do not move.

The next step is to consider charges that move.

While it is possible to consider the motion of single charges, in many situations there are too many single charges to keep track of. In order to treat the motion of many charges, we introduce a new vector field \( \mathbf{J}(\mathbf{r}, t) \) called the current density

\[
\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)
\]

where \( \rho(\mathbf{r}, t) \) is the number of charges per unit volume at \( \mathbf{r} \) at time \( t \).

\( \mathbf{v}(\mathbf{r}, t) \) is called the drift velocity. It is the average velocity of moving charges in a small region near \( \mathbf{r}, t \).

By definition, it is in the direction of moving positive charge. It points in the opposite direction for electrons.

Add these velocity vectors, divide by the number of particles. (-1) is the particle, all electrons.
For charge carriers with charge $q$

$$\bar{J} = q N_q \vec{v}_d$$

where $N_q$ is the number of particles with charge $q$ per unit volume and $\vec{v}_d$ is the drift velocity of those particles.

**Example**

Assume we have $10^{18}$ electrons per cubic centimeter moving with drift velocity $0.01 \text{ m/s}$ in the $x$ direction. Then

$$\bar{J} = (q_e) \frac{10^{18}}{(10^{-2})^3} \times 0.01 (-\hat{x})$$

Because the electrons have negative charge.

For this current consider

\[ A \hat{\vec{H}} \]

How much change passes through $A$ per second?
The total positive transported through \( dA \) is

\[
\Delta Q = \frac{(V_0 \Delta t)(dA_\perp)(\vec{n})}{\text{volume}} \text{ electrons} \times \frac{q_e}{\text{volume}}
\]

\[
= (V_0 \Delta t)(dA \cdot \vec{n}, \vec{v}_0)(\vec{n})(q_e)
\]

\[
= (V_0 \vec{v}_0) \cdot (\vec{n} dA) \cdot n q \Delta t
\]

\[
= \vec{v}_0 \cdot \vec{n} dA n q \Delta t
\]

\[
\frac{\Delta Q}{\Delta t} = (n q \vec{v}_0) \cdot (\vec{n} dA)
\]

\[
\frac{d\omega}{dt} = \vec{J} \cdot \vec{n} dA
\]

In this example \( n \) and \( \vec{v}_0 \) are constant. In general they may depend on \( \vec{x}, t \).

\[
\frac{d\omega}{dt} = \oint S \vec{J} \cdot \vec{n} dA = \text{is called the electric current through the area } A
\]
The unit of current is:

\[ 1 \text{ Ampere} = \frac{1 \text{ coulomb}}{\text{second}} \]

Recall - this is how the coulomb is defined.

If we break the area up into 2 parts \( A = A_1 + A_2 \)

\[
\oint \vec{J} \cdot \hat{n} \, dA = \oint \vec{J}_1 \cdot \hat{n} \, dA + \oint \vec{J}_2 \cdot \hat{n} \, dA
\]

\[ I = I_1 + I_2 \]

This is nothing more than saying charge can neither be created or destroyed.

What is the drift velocity of electrons in a copper wire of cross sectional area \((\text{mm})^2\) at a current of 1 Ampere?

(Assume \( n \) is uniform)

\[ I = \oint \vec{J} \cdot \hat{n} \, dA = (n \, V_0 \, \epsilon) \, (1 \text{mm})^2 \]

\[ V_0 = \frac{1 \text{ Ampere}}{(1\text{mm})^2 \, (9 \times 10^{-19}) \text{(coul)}} \]
\[ q_e = 1.6 \times 10^{-19} \, \text{C} \]

\[ n_{Cu} = \left( \frac{\text{atoms}}{\text{mol e}} \right) \left( \frac{\text{mol e}}{\text{kg}} \right) \left( \frac{\text{kg}}{\text{m}^3} \right) \]

\[ (6.02 \times 10^{23}) (63.54 \times 10^3) (8.96 \times 10^{-3} \times 10^6) \]

\[ = 2.42 \times 10^{32} \]

\[ V_0 = \frac{1}{10^{-6} (1.6 \times 10^{-19}) (2.42 \times 10^{32})} = 1.8 \times 10^{-8} \, \text{m/s} \]

which shows that the drift velocity through a copper wire at 1 ampere is still very slow.

In general, the current density will not be uniform.

\[ \mathbf{J}(\mathbf{r}, t) = \mathbf{j}(r) \hat{x} \]

\[ I = \int_0^r dr \int_0^{2\pi} r d\phi \mathbf{j}(r) \hat{x} \cdot \hat{x} = 2\pi \int_0^r r dr r \mathbf{j}(r) \]

\[ = 2\pi \int_0^r r dr (r \mathbf{e}_r) = 2\pi \left( \frac{r^2}{3} \right) \mathbf{e}_r \]

\[ I = \frac{2\pi}{3} \mathbf{e}_r \]

\[ \mathbf{e}_r \]

\[ \mathbf{e}_r \]

\[ \mathbf{e}_r \]

\[ \mathbf{e}_r \]
consider a collection of electrons in a constant electric field in the $x$ direction

$$\mathbf{E} = E\hat{x}$$

The force on an electron is

$$\mathbf{F} = q_e\mathbf{E} = q_eE\hat{x}$$

(where $q_e$ is negative). If the electron is in a medium, it will collide. The number of collisions per second will be approximately proportional to the velocity. This suggests that the net force is

$$\mathbf{F}_{\text{net}} = q_eE\hat{x} - c\mathbf{v}_d$$

The electron will accelerate until the force vanishes

$$\mathbf{v}_d = \frac{q_e\mathbf{E}}{c}$$

Using this in the definition of the current

$$\mathbf{J} = q_n\mathbf{v}_d = q_n\left(\frac{q_e\mathbf{E}}{c}\right) = q_n\left(\frac{q_e}{c}\right)\mathbf{E}$$

This shows that the current density is proportional to the electric field strength.
The relation
\[ V = IR \]
is called Ohm's law. The derivation followed from the assumption that the effect of collisions was due to a damping force proportional to the drift velocity.

Going back to the definition
\[ \vec{J} = qn \vec{V}_0 \]
\[ = qn \left( \frac{q}{\varepsilon} \vec{E} \right) \]
which shows that the current density is proportional to the electric field.

This is usually expressed as
\[ \vec{E} = \rho \vec{J} \]
where \( \rho \) is called the **resistivity** - which in this case only depends on the material. This relation can equivalently be expressed as
\[ \vec{J} = \sigma \vec{E} \]
This is written as

\[ \bar{J} = \sigma \bar{E} \]

where \( \sigma \) is called the **conductivity** of the material. \( \bar{J} \) can also be written

\[ \bar{E} = \frac{1}{\sigma} \bar{J} = \rho \bar{J} \]

\( \rho = \frac{1}{\sigma} \) is called the **resistivity** of the material.

Next consider a wire of cross sectional area \( A \) and length \( L \). Assume the ends are maintained at a potential \( V \).

\[ \bar{E} \cdot \bar{E} = V = \text{potential difference} \]

\[ I = \int \bar{J} \cdot \hat{n} \, dA = \int \sigma \bar{E} \cdot \hat{n} \, dA = \sigma \frac{V}{L} A = \left( \frac{\sigma A}{L} \right) V \]

This shows that the current is proportional to the potential difference.

\[ V = IR ; \quad R = \frac{\frac{1}{\sigma}}{\frac{1}{A}} = \rho \frac{L}{A} \]

\( R \) is called the **resistance**.
The unit of resistance is called 1 ohm = \frac{1 \text{ Volt}}{1 \text{ Amp}} = \frac{\text{ Joule}}{\text{ Coulomb} \cdot \text{ Coulomb}}.

The relation

V = IR

is called ohms law.

Note for a wire

\[ R = \rho \frac{L}{A} \]

which means the resistance increases with length, and decreases with increasing cross sectional area.

The resistivity generally increases with temperature.

\[ \rho = \rho_0 + \rho_0 \alpha (T - T_0) \]
\[ \rho_0 = \rho(T_0) \]

\( \alpha \) is called the temperature coefficient of resistivity.
Circuits involving resistors

\[ R \quad \text{notation for resistors} \]
\[ R_1 \quad R_L \quad \text{resistors in series} \]
\[ R_1 \quad \text{resistors in parallel} \]

Resistors in series:

\[ I_1 = I_2 = I \quad \text{(charge is conserved)} \]
\[ V_1 = IR_1, \quad V_2 = IR_L \]
\[ V = V_1 + V_2 = I (R_1 + R_L) = IR \]
\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_L} \quad \text{(resistors in series)} \]

Resistors in parallel:

\[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_L} \]
\[ R = \frac{R_1 R_L}{R_1 + R_L} \quad \text{(resistors in parallel)} \]
These rules are the opposite of the corresponding rules in capacitive.

\[ R_{34} = \frac{R_3 R_4}{R_3 + R_4} \quad R_{134} = R_1 + \frac{R_3 R_4}{R_3 + R_4} \]

\[ R = \frac{1}{R_2} + \frac{1}{R_1 + \frac{R_3 R_4}{R_3 + R_4}} \]

\[ \frac{1}{R} = \frac{R_2 \left( R_1 + \frac{R_3 R_4}{R_3 + R_4} \right)}{R_1 + R_2 + \frac{R_3 R_4}{R_3 + R_4}} = \frac{R_2 \left( R_1 R_3 + R_1 R_4 + R_3 R_4 \right)}{R_1 R_3 + R_2 R_4 + R_3 R_4} \]

\[ V = I R \]

\[ I_2 = \frac{V}{R_2} \]

\[ I_1 = \frac{V}{R_{134}} = \frac{V}{R_1 + \frac{R_3 R_4}{R_3 + R_4}} \]

\[ V_1 = I_1 R \]

\[ I_3 = \frac{V - V_1}{R_3} \]

\[ I_4 = \frac{V - V_1}{R_4} \]