Lecture 16

Multiloop circuits

$V = IR$ ohms Law

Resistors in series

$$R = \sum_{i=1}^{N} R_i$$

Resistors in parallel

$$\frac{1}{R} = \sum_{i=1}^{N} \frac{1}{R_i}$$

Power Dissipated

$$P = IV = I^2 R = V^2/c$$

Ideal EMF device

$\mathcal{E}$

does work

key concepts

① charge conservation

* net current into any junction is 0

② energy conservation

* work done moving a charge to is original position is 0 ⟹ potential does not change around a closed loop

Steps

① draw circuit

Resistors EMF Capacitors Switches

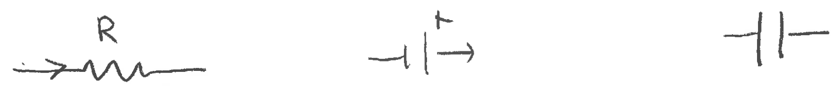
label components

Draw arrows indicating guessed current direction

label currents

② require current conservation at each vertex

require potential conservation around any closed loop



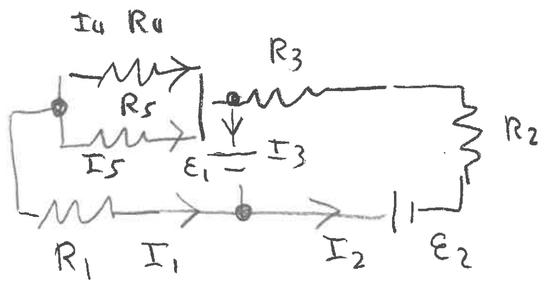
$V = IR$ decreases in direction of arrow

$V = \varepsilon$ increases in direction of arrow

$V = Q/C$

③ determine number of unknowns

construct equations using loops and vertex equations



vertex equations:

$$I_1 + I_3 = I_2$$

$$I_2 + I_4 + I_5 = I_3$$

$$0 = I_1 + I_4 + I_5$$

loop equations:

$$-R_5 I_5 + R_4 I_4 = 0$$

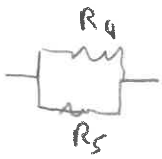
$$I_5 R_5 - I_1 R_1 - \mathcal{E}_1 = 0$$

$$\mathcal{E}_2 - I_2 R_2 - I_3 R_3 + \mathcal{E}_1 = 0$$

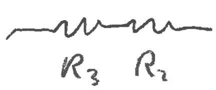
there are additional loop equations
this gives a system of 5 linear
equations that can be solved
for the currents

these can be put in the form of
a matrix equation that can be
solved using methods of linear
algebra.

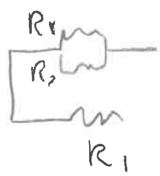
In this case we can replace



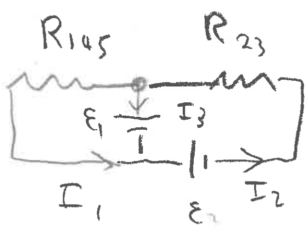
$$R_{45} = \frac{R_4 R_5}{R_4 + R_5}$$



$$= \quad R_{32} = R_2 + R_3$$



$$= \quad \frac{R_{45} R_1}{R_1 + R_{45}}$$



$$\mathcal{E}_1 - I_1 R_{145}$$

$$- \mathcal{E}_1 - \mathcal{E}_2 - I_2 R_3 = 0$$

$$I_1 + I_3 = I_2$$

$$I_1 = \frac{\mathcal{E}_1}{R_{145}}$$

$$I_2 = -\frac{(\mathcal{E}_1 + \mathcal{E}_2)}{R_3}$$

$$I_3 = -\frac{(\mathcal{E}_1 + \mathcal{E}_2)}{R_3} - \frac{\mathcal{E}_1}{R_{145}}$$

given these we use

$$I_5 = (\mathcal{E}_1 + I_1 R_1) / R_5$$

$$I_4 = \frac{R_5}{R_4} I_5 = (\mathcal{E}_1 + I_1 R_1)/R_4.$$

so we can get some simplification using combined circuit elements

EMF devices

① convert chemical or mechanical energy into electrical energy

this in turn can be converted to energy in a device

the power is the rate of change of energy.

$$dW = V\,dq$$

$$\frac{dW}{dt} = V\frac{dq}{dt} = VI = P$$

combining this with ohms law

$$P = VI = I^2R = V^2/R$$

consider a 100 watt light bulb →

$$VI = 100 = V^2/R = (110)^2/R$$

$$\boxed{R = 121\ \Omega}$$

next consider the power dissipated by 2 light bulbs in series or parallel.

series: for fixed potential differenc

$$R = 2R_1$$

$$P = V^2/R \rightarrow V^2/2R$$

parallel

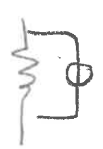
$$R = \frac{RR}{R+R} = \frac{1}{2}R$$

$$P = V^2/(\tfrac{1}{2}R) = 2V^2/R$$

By putting the bulds in parallel 4 time as much power is dissipated than if they are in series – in fact 2 bulbs in series use less power that one in series.

This is why home wiring is in parallel.

Meters: Volts Amps



For a volt meter the goal is to minimize the current through the meter

$$R_{\parallel} = \frac{R R_M}{R_M + R} = \frac{R}{1 + R/R_m}$$

$(R_M >> R \Rightarrow R_{\parallel} \sim R$ so IR does not we treat this using energy conservation

$$\mathcal{E} - \frac{Q}{C} - IR = 0$$ (change in potential around a loop = 0)

recall that $I = \frac{dQ}{dt}$. using this in the above equation gives

$$\boxed{R\frac{dQ}{dt} + \frac{Q}{C} - \mathcal{E} = 0}$$

this is called a differential equation

this can be solved using

$$\frac{dQ}{dt} = \frac{1}{R}\left(\mathcal{E} - \frac{Q}{C}\right)$$

we write this as

$$dQ = \frac{1}{R}\left(\mathcal{E} - \frac{Q}{C}\right)dt$$

$$dt = \frac{dQ}{\frac{1}{R}\left(\mathcal{E} - \frac{Q}{C}\right)}$$

we assume $Q(t=0) = 0$; then

$$\int_0^t dt = \int_0^{Q(t)} \frac{dQ}{\frac{1}{R}\left(\mathcal{E} - \frac{Q}{C}\right)}$$

Let $u = \frac{1}{R}(\varepsilon - \frac{Q}{C})$

$du = -\frac{1}{RC}\, dQ$

$dQ = -RC\, du$

at $t = 0$ $Q = 0$ $u = \frac{\varepsilon}{R}$

$$\int_0^t = \int_{\varepsilon/R}^{u(t)} \left(-RC \frac{du}{u}\right) = -RC \ln\left(\frac{u}{\varepsilon/R}\right) = t$$

$$\ln\left(\frac{u}{\varepsilon/R}\right) = -\frac{t}{RC}$$

$$e^{\ln\left(\frac{u}{\varepsilon/R}\right)} = e^{-t/RC}$$

$$u = \frac{\varepsilon}{R} e^{-t/RC} = \frac{\varepsilon}{R} - \frac{Q}{RC}$$

$$\frac{Q}{RC} = \frac{\varepsilon}{R}\left(1 - e^{-t/RC}\right)$$

$$\boxed{Q(t) = C\varepsilon\left(1 - e^{-t/RC}\right)}$$

This indicates that the charge starts at 0, and eventually reaches $C\varepsilon$.

We can also determine the current in the resistor

$$I(t) = \frac{dQ}{dt} = C\mathcal{E}\left(-\left(-\frac{1}{RC}\right)\right) e^{-t/RC}$$

$$I(t) = \frac{\mathcal{E}}{R} e^{-t/RC}$$

In this case the current starts at $\frac{\mathcal{E}}{R}$ and exponentially goes to 0.

Since t/RC must be dimensionless then $RC = \tau$ has dimensions of time. It is call the time constant - it increases with increasing R a C.

We can also consider discharging a capacitor. In this case $\mathcal{E} = 0$

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

This is the same equation with $\mathcal{E} = 0$. In this case the initial $Q = Q_0$

$$\frac{dQ}{Q} = -\frac{dt}{RC}$$

integrating

$$\int_{Q_0}^{Q} \frac{dQ'}{Q'} = \int_0^t -\frac{dt}{RC}$$

$$\ln Q - \ln Q_0 = -t/RC$$

$$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC}$$

$$e^{\ln\left(\frac{Q}{Q_0}\right)} = e^{-t/RC}$$

$$\boxed{Q = Q_0 e^{-t/RC}}$$

as t gets large the charge on the capacitor decrease to 0. The current is

$$I = \frac{dQ}{dt} = -\frac{Q_0}{RC} e^{-t/RC}$$

The power $\quad V = Q/C$

$$P = IV = \left(-\frac{Q_0}{RC} e^{-t/RC}\right)\left(\frac{Q_0}{C} e^{-t/RC}\right)$$

$$P = -\frac{Q^2}{RC^2} e^{-2t/RC}$$

we can use this to find the energy stored in the capacitor

$$E = \int_0^\infty P\,dt = -\frac{Q^2}{RC^2}\int_0^\infty e^{-2t/RC}$$

$u = 2t/RC$

$du = \frac{2}{RC}\,dt$

$dt = \frac{RC}{2}\,du$

$$= -\frac{Q_0}{RC^2}\cdot\frac{RC}{2}\int_0^\infty e^{-u}\,du$$

$$= -\frac{Q_0^2}{C}\left(-e^{-\infty} - -e^{-0}\right)$$

$$= -\frac{1}{2}\frac{Q_0^2}{C}$$

This shows that the energy lost by the capacitor is exactly the stored energy. For small RC it discharges quickly problem Leaky capacitor

$C = 2\mu F \qquad Q(0) = Q_0 \qquad Q(2s) = \frac{1}{4} Q_0$

$Q = Q_0 e^{-t/RC}$

$\frac{1}{4}\cancel{Q_0} = \cancel{Q_0} e^{-t/RC}$

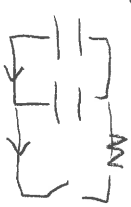
$\ln \frac{1}{4} = -t/RC$

$0 - \ln 2 = -t/RC$

$\boxed{R = \frac{t}{C \ln 2}} \qquad \frac{2s}{2\times10^{-6} \ln 2} =$

capacitors in parallel

C_1



C_2

assume these start fully charged

loop 1 $\qquad C = C_1 + C_2$

$Q(t) = Q_0 e^{-t/R(C_1+C_2)}$

$Q = Q_1 + Q_2 \qquad V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \qquad Q_2 = \frac{C_2}{C_1} Q_1$

$= Q_1 \left(1 + \frac{C_2}{C_1}\right)$

$Q_1(t) = \frac{1}{(1 + C_2/C_1)} Q_0 e^{-t/R(C_1+C_2)} = \frac{Q_0 C_1}{C_1 + C_2} e^{-t/R(C_1+C_2)}$

$Q_2(t) = \frac{Q_0 C_2}{C_1 + C_2} e^{-t/R(C_1+C_2)}$

$$I_1 = -\frac{Q_0 C_1}{R} e^{-t/R(C_1+C_2)}$$

$$I_2 = \frac{Q_0 C_2}{R} e^{-t/R(C_1+C_2)}$$