Lecture 19

Magnetic force on current carrying wire

* $\bar{F} = q\,\bar{V} \times \bar{B}$

q = charge

$\bar{V}$ = particle velocity

$\bar{B}$ = magnetic field

Field units

1 Tesla = (N · sec)/(coulomb · meter)

A lenght dl of wire contains many charge carriers, the superposition principle gives

$$\bar{F} = \sum_i q_i\,\bar{V}_i \times \bar{B}(\bar{r}_i, t)$$

this is the sum of the forces on every charge in the wire

Let $d\bar{l}$ be a vector of length dl pointing in the direction of the current

This is the same direction as

$\bar{V}_D \cdot q$

For a small enough $d\bar{l}$ $\bar{B}(rt)$ is approximately constant

$$d\bar{F} = \sum q_i \bar{V}_i \times \bar{B}$$

we replace volume

$$\sum q_i \bar{V}_i = n e \vec{V}_D A \cdot dl$$

since $d\bar{l} \propto e\bar{V}_D$

$$d\bar{F} = n l e \bar{V} l A \cdot d\bar{l} \times \bar{B}$$

$$= I \, d\bar{l} \times \bar{B}$$

This means that the total force on the wire is

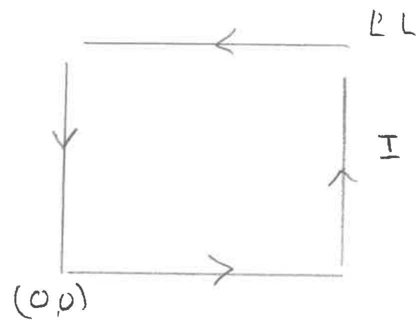
$$\bar{F}_T = \sum I \, d\bar{l}_i \times \bar{B}(\bar{r}_i)$$

which in the limit of small $d\bar{l}_i$ gives

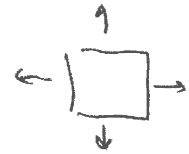
$$\bar{F} = \int I \, d\bar{l} \times \bar{B}$$

where the integral is along the path of the wire example 1

consider a square loop of wire with current I and side L in the xy plane. Let $\bar{B}$ be a constant magnetic field



for $B = \hat{z} B$



$$\oint I\, d\bar{l} \times \bar{B} = \int_0^L dx\, I\, \hat{x} \times \bar{B} + \int_0^L dy\, I\, \hat{y} \times \bar{B} + \int_L^0 dx\, I\, \hat{x} \times \bar{B} \; \int_L^0 dy\, I\, \hat{y} \times \bar{B}$$

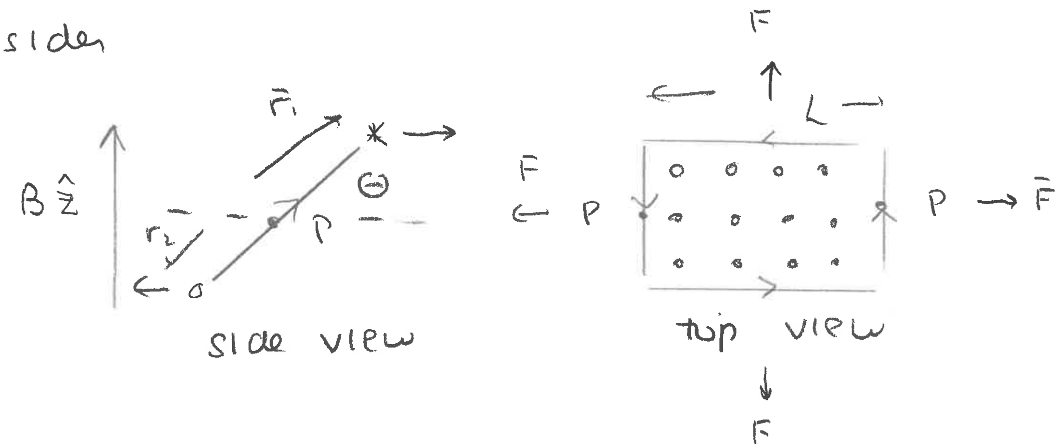
$$= IL(\hat{x} + \hat{y} - \hat{x} - \hat{y}) \times \bar{B}$$

$$= 0$$

This shows that the field does no work on this current loop. While there is a net force on each side, the forces all cancel

The forces can exert a torque on the current loop

Consider



Looking at the picture it is clear that the magnetic forces on this loop are producing a torque

The relevant forces are the ones in and out of the plane of the paper in the figure on the right

$$ILB(-\hat{x} \times \hat{z}) = ILB\hat{y} \quad *$$

$$ILB(\hat{x} \times \hat{z}) = -ILB\hat{y} \quad \odot$$

The torque about P

$\bar{r}_1 = \frac{L}{2}(\hat{y}\cos\theta + \hat{z}\sin\theta)$

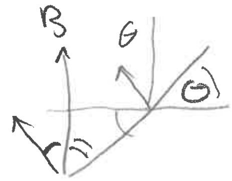
$\bar{r}_2 = -\frac{L}{2}(\hat{y}\cos\theta + \hat{z}\sin\theta)$

$$\bar{r}_1 \times \bar{F}_1 + \bar{r}_2 \times \bar{F}_2 =$$

$$-\frac{L}{2}\sin\theta(ILB)\hat{x} + \left(-\frac{L}{2}\sin\theta\right)(-ILB)\hat{x}$$

$$-IL^2B\sin\theta\,\hat{x}$$

In this case $L^2 = A$ = area of current loop



$$\boxed{T = IA\,\hat{n} \times \bar{B}}$$

recall for an electric dipole the torque is

$$\boxed{T = \bar{p} \times \bar{E}}$$

where $\bar{p}$ is the dipole moment

we see that for a square loop $IA\hat{n}$ behaves like a dipole with respect to the magnetic field

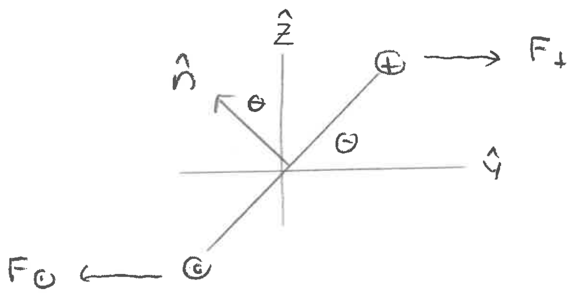
the work needed to rotate the dipole against the torque

we define the magnetic dipole moment of this current loop

$$\bar{\mu} = IA\hat{n}$$

$$T = \bar{\mu} \times \bar{B}$$

We can also compute the potential energy of this dipole in the magnetic field $\bar{B}$



$F_+ = IBL\hat{y}$

$F_\circ = -IBL\hat{y}$

Let $\bar{r}_+ = \frac{L}{2}(\hat{y}\cos\theta + \hat{z}\sin\theta)$

$\bar{r}_- = \frac{L}{2}(-\hat{y}\cos\theta - \hat{z}\sin\theta)$

$d\bar{r}_+ = \frac{L}{2}(-\hat{y}\sin\theta + \hat{z}\cos\theta)\,d\theta$

$d\bar{r}_- = \frac{L}{2}(\hat{y}\sin\theta - \hat{z}\cos\theta)\,d\theta$

work done against the field by increasing θ is

$$dW = -\bar{F}\cdot d\bar{r} = -\bar{F}_+ \cdot d\bar{r}_+ - \bar{F}_- \cdot d\bar{r}_-$$

$$= -(IBL\hat{y})\cdot\left(-\tfrac{L}{2}\hat{y}\sin\theta + \tfrac{L}{2}\hat{z}\cos\theta\right)d\theta + $$
$$-(-IBL\hat{y})\cdot\left(\tfrac{L}{2}\hat{y}\sin\theta - \tfrac{L}{2}\hat{z}\cos\theta\right)d\theta$$

$$= \frac{L^2IB}{2}(\sin\theta + \sin\theta)\,d\theta$$

$$= L^2IB\sin\theta = IAB\sin\theta$$

Integrating gives

$$\int (-F \cdot dr) = \int_{\theta_0}^{\theta} IAB \sin\theta \, d\theta$$

$$- IAB \cos\theta + IAB \cos\theta_0$$

The constant is irrelevant

$$U = - IAB \cos\theta$$

but $\mu = IA\hat{n}$ $\quad B = B\hat{z}$ $\quad \hat{n} \cdot \hat{z} = \cos\theta$

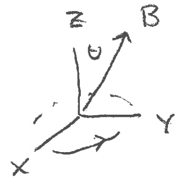
$$U = - \vec{\mu} \cdot \overline{B}$$

which looks similar to $-\overline{p} \cdot \overline{E} = U$

In this sense we can understand current loops as magnetic dipoles

It turns out that this result is the same for any shape.

Case of a circular current loop



$\bar{B}$ in $\hat{y}\ \hat{z}$ plane

Loop in x y plane

$$d\bar{l} = d(R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y})$$
$$= (-R\sin\phi\,\hat{x} + R\cos\phi\,\hat{y})\,d\phi$$

consider torque about x axis

$$\bar{B} = B\hat{y}\sin\theta + B\hat{z}\cos\theta$$

the force on a section of wire

$$dF = I\,d\bar{l}\times\bar{B}$$
$$= I(-R\sin\phi\,\hat{x} + R\cos\phi\,\hat{y})\,d\phi \times (B\hat{y}\sin\theta + B\hat{z}\cos\theta)$$
$$= IRB(-\hat{z}\sin\theta\sin\phi + \hat{x}\cos\phi\cos\theta + \hat{y}\sin\phi\cos\theta)$$

If we integrate around the circle all 3 terms vanish

If we want the torque about the $\hat{x}$ axis

$$\bar{\tau} = \int \bar{r}\times d\bar{F} =$$
$$= (R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y})\times IRB(-\hat{z}\sin\phi\sin\theta + \hat{x}\cos\phi\cos\theta + \hat{y}\sin\phi\cos\theta)$$
$$= \hat{y}\,R^2IB\cos\phi\sin\phi\sin\theta + \hat{z}\,R^2IB\cos\phi\sin\phi\cos\theta$$
$$-\hat{x}\,R^2IB\sin^2\phi\sin\theta - \hat{z}\,R^2IB\sin\phi\cos\phi\cos\theta$$

$\sin\phi \cos\phi = \frac{1}{2}\sin 2\phi \qquad u = 2\phi \qquad du = 2d\phi$

$\int_0^{2\pi} \frac{1}{2}\sin 2\phi \, d\phi = \frac{1}{4}\int_0^{4\pi} \sin u \, du = -\frac{1}{4}(\cos 4\pi - \cos 0) = 0$

The only term that contributes is

$-\hat{x} R^2 I B \int_0^{2\pi} \sin^2\phi \sin\theta$

The integral $\int_0^{2\pi} \sin^2\phi = \frac{1}{2}\int_0^{2\pi} (\sin^2\phi + \cos^2\phi) = \pi$

$\bar{\tau} = -\pi R^2 I B \sin\theta \, \hat{x}$

Note: $\hat{n} \times \bar{B} = \hat{z} \times (B\hat{y}\sin\theta + B\hat{z}\cos\theta) = -B\hat{x}\sin\theta$

This gives the torque

$$\bar{\tau} = \underbrace{(\pi R^2)}_{A} I \, \hat{n} \times \bar{B}$$

$$= I A \hat{n} \times \bar{B}$$

$$= \bar{\mu} \times \bar{B}$$

where $\bar{\mu} = I A \hat{n}$

Consider a "hydrogen atom" with an electron in a circular orbit around a proton

$\mu = IA = \frac{q}{T} \cdot \pi r^2 \qquad r = 5.29 \times 10^{-11}$

$\frac{mv^2}{r} = \frac{kq^2}{r^2} \qquad v = r\omega \qquad \omega = 2\pi f$

so far what we have done is the analog of $\bar{F} = q\bar{E}$ for magnetic fields. The thing that is missing is the analog of Coulomb's law or Gauss law for magnets.

The relevant law is called the Biot - Savart Law

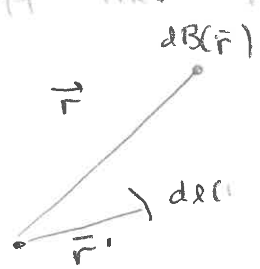
It has the form

$$d\bar{B} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{l} \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

which is the contribution to the field B at $\bar{r}$ due to a current at $\bar{r}'$

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{T m}{A}$$

Like coulomb law the field strength falls off like $1/r^2$



we can use the superposition principle to find the full field at $\bar{r}$ by adding up all of the contributions in a wire

$$\bar{B} = \int_{wire} dB = \frac{\mu_0 I}{4\pi} \int \frac{d\bar{l}' \times (\bar{r} - r')}{|\bar{r} - \bar{r}'|^3}$$

this is considerably more complicated than coulombs law, some special cases are not too difficult

case 1

$\bar{B}$ due to a long straight wire along the z axis with current I

$I\, d\bar{l} = I\, dz\, \hat{z}$ $\qquad \bar{r}' = z\hat{z} = (0\ 0\ z)$

$d\bar{l}' \times (\bar{r} - \bar{r}') = dz' \hat{z} \times (x\hat{x} + y\hat{y} + (z - z')\hat{z}) =$

$dz' (x\hat{y} - y\hat{x})$

$|r - r'|^2 = x^2 + y^2 + (z - z')^2$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} dz' \; \frac{x\hat{y} - y\hat{x}}{\left(x^2+y^2+(z-z')^2\right)^{3/2}}$$

let $\quad z'' = z' - z \qquad dz'' = dz' \qquad z'' ; -\infty \to \infty$

$$\vec{B}(x\,y\,z) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} dz'' \; \frac{x\hat{y} - y\hat{x}}{(x^2+y^2+z''^2)^{3/2}} \, dz''$$

$$= \frac{\mu_0 I}{4\pi} (x\hat{y} - y\hat{x}) \int_{-\infty}^{\infty} \frac{dz''}{(r^2+z''^2)^{3/2}} \qquad r^2 = x^2+y^2$$

To do the integral note

$$\frac{d}{dz}\left(z(z^2+r^2)^{-1/2}\right) = (z^2+r^2)^{-1/2} + z\left(-\tfrac{1}{2}\right)(z^2+r^2)^{-3/2} \cdot (2z)$$

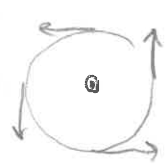
$$= \frac{z^2+r^2}{(z^2+r^2)^{3/2}} - \frac{z^2}{(z^2+r^2)^{3/2}} = \frac{r^2}{(z^2+r^2)^{3/2}}$$

$$\therefore \quad \frac{1}{(z^2+r^2)^{3/2}} = \frac{1}{r^2} \frac{d}{dz}\left(\frac{z}{(z^2+r^2)^{1/2}}\right)$$

using this in the above gives

$$\vec{B}(x\,y\,z) = \frac{\mu_0 I}{4\pi}(x\hat{y} - y\hat{x}) \frac{1}{r^2} \underbrace{\left.\frac{z}{(z^2+r^2)^{1/2}}\right|_{-\infty}^{\infty}}_{2}$$

$$\boxed{\vec{B}(x\,y\,z) = \frac{\mu_0 I}{2\pi} \frac{x\hat{y} - y\hat{x}}{r^2}}$$



This satisfies the right hand rule in the sense that following the field lines with the right hand rule gives the direction of the current