Lecture 20

Magnetic dipoles

$\bar{\mu}$ = magnetic moment

$= I A \hat{n}$

I = current in a loop

A = area inside of the loop

$\hat{n}$ = normal to the area - in the direction using the right hand rule with respect to current direction

We showed to a disk and square

$\mathcal{U} = -\bar{\mu} \cdot \bar{B}$ (potential energy)

$\bar{\tau} = \bar{\mu} \times \bar{B}$ (torque)

These calculations were exact, unlike the calculations for electric dipoles, which only became exact in the limit that $d \to 0$ and $q \to \infty$ keeping $qd = p$ constant.

example - classical hydrogen atom. - assume a circular orbit - electron around proton.

the current is $q_e \times \frac{1}{T}$ where T is the period of the orbit. For central force motion

$$\frac{mv^2}{R} = kq^2 \frac{1}{R^2}$$

$$\omega = \frac{V}{R} = 2\pi f = \frac{2\pi}{T}$$

$$\frac{m R^2 \omega^2}{R} = m R \omega^2 = m R \left(\frac{2\pi}{T}\right)^2 = \frac{kq^2}{R^2}$$

$$\frac{1}{T^2} = \frac{kq^2}{4\pi^2 m R^3}$$

$$I = \frac{q}{T} = q^2 \sqrt{\frac{k}{4\pi^2 m R^3}} = \frac{q^2}{2\pi} \sqrt{\frac{k}{mR^3}}$$

the area is πR^2

$$\mu = \frac{q^2}{2\pi} \left(\sqrt{\frac{k}{mR^3}} \right) \times \pi R^2 =$$

$$= \frac{q^2}{2} \sqrt{\frac{k}{m} R}$$

where R can be taken as the Bohr radius of the Hydrogen atom.

this explains the source of magnetic dipoles

The key property of magnetic dipoles is that they reach a lowest energy state when $\vec{\mu}$ is aligned parallel to the magnetic field

permanent magnetics are nothing more than a bunch of microscopic dipoles that are all lined up.

dipoles respond to the presence of a magnetic field (like a compass)
the next step is to understand the source of magnetic fields

our demonstration showed that magnets have fields - this suggests that currents are a source of magnetic fields

the analog on Coulombs law for magnetic fields is called the Biot-Savart law

$$\boxed{d\bar{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\, d\bar{\ell} \times (\bar{r}-\bar{r}')}{|r-r'|^3}}$$

this is proportional to the current and falls off like $1/r^2$

To compute the force due to current in a wire it is necessary to integrate along the length of the wire

$$\mu_0 = 4\pi \times 10^{-7} \quad \frac{\text{Tesla}}{C} \quad \frac{S}{m^2}$$

example: magnetic field due to an infinite wire with current I

$$d\bar{l} = \hat{z}\, dz' \qquad z; -\infty \;\; \infty$$

$$\bar{r}' = z\hat{z}$$

using the Biot Savart law gives

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dz'\, \hat{z} \times (\bar{r} - z'\hat{z})}{|\bar{r} - z\hat{z}|^3}$$

note $\bar{r} - z'\hat{z} = \hat{x}x + \hat{y}y + (z-z')\hat{z}$

$$|r - z'\hat{z}|^2 = x^2 + y^2 + (z-z')^2$$

note $\hat{z} \times (\bar{r} - z'\hat{z}) = x\hat{y} - y\hat{x}$ (both constant)

$$\bar{B} = \frac{\mu_0 I}{4\pi}(x\hat{y} - y\hat{x}) \int_{-\infty}^{\infty} \frac{dz'}{(x^2+y^2+(z-z')^2)^{3/2}}$$

Let $z'' = z' - z$, then this becomes

$$\bar{B} = \frac{\mu_0 I}{4\pi}(x\hat{y} - y\hat{x}) \int_{-\infty}^{\infty} \frac{dz''}{(r^2 + z''^2)^{3/2}} \qquad (r^2 \equiv x^2 + y^2)$$

to do the integral note

$$\frac{d}{dz}\left(\frac{z}{\sqrt{r^2+z^2}}\right) = \frac{1}{\sqrt{r^2+z^2}} + z\left(-\frac{1}{2}\right)\frac{1}{\left(\sqrt{z^2+r^2}\right)^3}(2z)$$

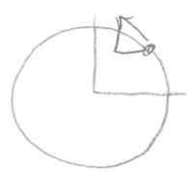
$$= \frac{r^2+z^2-z^2}{\left(\sqrt{r^2+z^2}\right)^{3/2}} = \frac{r^2}{\left(\sqrt{r^2+z^2}\right)^3}$$

this means

$$\frac{1}{\left(\sqrt{r^2+z^2}\right)^3} = \frac{1}{r^2}\frac{d}{dz}\left(\frac{z}{\sqrt{r^2+z^2}}\right)$$

$$\therefore \quad B = \frac{\mu_0 I}{4\pi}(x\hat{y} - y\hat{x})\frac{1}{r^2}\underbrace{\left\{\frac{z}{\sqrt{r^2+z^2}}\right\}_{-\infty}^{+\infty}}_{2}$$

$$\boxed{\bar{B} = \frac{\mu_0 I}{2\pi}\,\frac{x\hat{y} - y\hat{x}}{r^2}}$$



we can that the field lines are circular, falling off like $\frac{1}{r^2}$

example 2 field due to act at center of circle

$$\bar{r} = (\hat{x}\cos\theta + \hat{y}\sin\theta)R$$

$$d\bar{l} = ((-\hat{x}\sin\theta + \hat{y}\cos\theta)\,d\theta)R$$

$$\bar{B}(0) = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{(-\hat{x}\sin\theta + \hat{y}\cos\theta)\,d\theta \times (-\hat{x}\cos\theta - \hat{y}\sin\theta)\,R^2}{R^3}$$

$$= \frac{\mu_0 I}{4\pi R}\int \hat{z}(\sin^2\theta + \cos^2\theta)$$

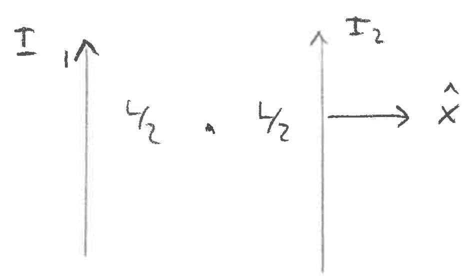
$$\bar{B}(0) = \frac{\mu_0 I}{4\pi R}\cdot \hat{z}\,(\theta_2 - \theta_1)$$

For a full loop

$$\bar{B}(0) = \frac{\mu_0 I}{4\pi R}\cdot 2\pi\,\hat{z} = \frac{\mu_0 I}{2R}$$

example 3 - superposition

case of 2 parallel wires. In this case we use the superposition principle. Both fields are in the x y plane for currents along the z axis



The magnetic field due to the first current is

$$\bar{B}_1 = \frac{\mu_0 I_1}{2\pi}\cdot \frac{(x+\frac{L}{2})\,\hat{y} - y\,\hat{x}}{(x+\frac{L}{2})^2 + y^2}$$

$$\bar{B}_2 = \frac{\mu_0 I_2}{2\pi}\,\frac{(x-\frac{L}{2})\,\hat{y} - y\,\hat{x}}{(x-\frac{L}{2})^2 + y^2}$$

adding these

$$\bar{B} = (\bar{B}_1 + \bar{B}_1) =$$

$$\frac{\mu_0}{2\pi}\left(\hat{y}\left(\frac{I_1(x+\frac{L}{2})}{r^2+\frac{L^2}{4}+xL} + \frac{I_2(x-\frac{L}{2})}{r^2+\frac{L^2}{4}-xL}\right)\right.$$

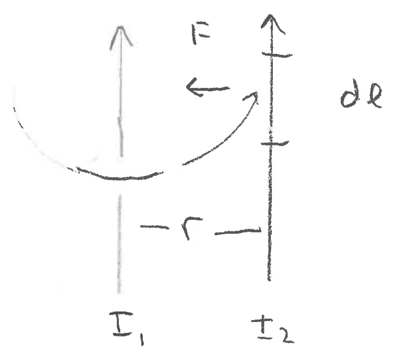
$$\left. -\hat{x}y\left(\frac{I_1}{r^2+\frac{L^2}{4}+xL} + \frac{I_2}{r^2+\frac{L^2}{4}-xL}\right)\right)$$

When there are 2 wires, while the both are sources of magnetic field, the field due to each wire exerts a force on the other.

$$\bar{B} = \frac{\mu_0 I}{2\pi}\left(\frac{\hat{\theta}}{r}\right)$$

$$\bar{F} = I\, d\bar{l} \times \bar{B}$$

$$\bar{F} = \frac{\mu_0 I_1 I_2}{2\pi}\frac{dl}{r}(-\hat{x})$$



∴ If the currents are in the same direction, then the wires are attracted; if they are in opposite direction they repel.

Magnetic field due to a point dipole

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int \frac{i\, d\bar{l} \times (\bar{r}-\bar{r}')}{|r-r'|^3}$$

$$\bar{r}' = R\hat{x}\cos\phi + R\hat{y}\sin\phi$$

$$d\bar{l} = -R\hat{x}\sin\phi + R\hat{y}\cos\phi$$

$$|\bar{r}-\bar{r}'|^2 = (x-R\cos\phi)^2 + (y-R\sin\phi)^2 + z^2$$

$$= (x^2+y^2+z^2) + R^2 - 2R(x\cos\phi + y\sin\phi)$$

as in the case of an electric dipole

we assume $r \gg R$

$$|\bar{r}-\bar{r}'|^2 = r^2 + R^2 - 2\bar{r}\cdot\bar{r}'$$

$$= r^2\left(1 - 2\frac{\bar{r}\cdot\bar{r}'}{r^2} + \frac{R^2}{r^2}\right)$$

$$|\bar{r}-\bar{r}'|^{-3} = r^{-3}\left(1-2\frac{\bar{r}\cdot\bar{r}'}{r^2} + \frac{R^2}{r^2}\right)^{-3/2}$$

$$= r^{-3}\left(1 - \frac{3}{2}(\quad)^{-5/2}\left(-2\frac{\bar{r}\cdot\bar{r}'}{r^2} + \frac{R^2}{r^2}\right) + O\left(\frac{R^2}{r^2}\right)\right)$$

this term falls off like R/r

keeping only the leading term

$$= \frac{1}{r^3}\left(1 + 3\frac{\bar{r}\cdot\bar{r}'}{r^2} + \cdots\right)$$

$$(y - R\sin\phi)\hat{y} + z\hat{z})$$

next note

$$d\bar{\ell} \times (\bar{r}-\bar{r}') =$$

$$(-R\hat{x}\sin\phi + R\hat{y}\cos\phi) \times ((x-R\cos\phi)\hat{x} + (y-R\sin\phi)\hat{y} + z\hat{z}) =$$

$$(R\sin\phi\,(y-R\sin\phi)\,\hat{z} + R\sin\phi\, z\,\hat{y}$$
$$(-R\cos\phi\,(x-R\cos\phi)\,\hat{z} + R\cos\phi\, z\,\hat{x} =$$

$$(R^2(\sin^2\phi+\cos^2\phi) + R(y\sin\phi + x\cos\phi)\;\hat{z}$$
$$+ Rz(\sin\phi\,\hat{y} + \cos\phi\,\hat{x})$$

this gets multiplied by

$$\frac{1}{r^3}\left(1 + 3\,\frac{\bar{r}\cdot\bar{r}'}{r^2} + \dots\right) =$$

$$\frac{1}{r^3}\left(1 + 3\,\frac{R}{r}\left(\frac{x}{r}\cos\phi + \frac{y}{r}\sin\phi\right) + \cdot\right.$$

when we do the integral over ϕ

$$\int_0^{2\pi} \sin\phi\, d\phi = \int_0^{2\pi} \cos\phi\, d\phi = 0$$

$$\int_0^{2\pi} \sin^2\phi\, d\phi = \int_0^{2\pi} \cos^2\phi\, d\phi = \pi$$

$$\int_0^{2\pi} \sin\phi\cos\phi = \frac{1}{2}\int_0^{2\pi} \sin 2\phi = 0$$

$$\int_0^{2\pi} \sin^3\phi\, d\phi = \int_{-\pi}^{\pi} \sin^3\phi\, d\phi = 0 \qquad (\sin^3(-\phi) = -\sin^3\phi)$$

$$\int_0^{2\pi} \cos^3\phi\, d\phi = \int_0^{\pi} \cos^3\phi + \int_{\pi}^{2\pi} \cos^3\phi = \int_0^{\pi} (\cos^3\phi - \cos^3\phi) = 0$$

$$\int_0^{2\pi} \cos^2\phi\sin\phi = \int_0^{2\pi} (1-\sin^2\phi)\sin\phi = 0$$

$$\int_0^{2\pi} \sin^2\phi\cos\phi = \int_0^{2\pi} (1-\cos^2\phi)\cos\phi = 0$$

of all of these terms only the terms with $\sin^2\phi$ or $\cos^2\phi$ contribute.

we write out the terms multiplying $\cos^2\phi$ & $\sin^2\phi$

$$\frac{\mu_0 I}{4\pi r^3}\iint R^2\cos^2\phi\,\hat{z} + R^2\sin^2\phi\,\hat{z}$$
$$+ 3\frac{R^2}{r^2}\left(-x^2\cos^2\phi\,\hat{z} + xz\cos^2\phi\,\hat{x} - y^2\sin^2\phi\,\hat{z} + yz\sin^2\phi\,\hat{y}\right] d\phi$$

$$\frac{\mu_0 I}{4\pi r^3}\left[2\pi R^2\hat{z} + 3\frac{\pi R^2}{r^2}\left(-x^2-y^2-z^2\right)\hat{z} + z^2\hat{z} + yz\hat{y} + xz\hat{x}\right)$$

$$I\pi r^2\hat{z} = \bar{\mu}$$

$$\bar{B} = \frac{\mu_0}{4\pi}\cdot\frac{1}{r^3}\left[2\bar{\mu} + 3\left(-\bar{\mu} + \frac{\bar{\mu}\cdot\bar{r}\,\bar{r}}{r^2}\right)\right]$$
$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\frac{\bar{\mu}\cdot\bar{r}\,\bar{r}}{r^2} - \bar{\mu}\right] = \frac{\mu_0}{4\pi r^5}\left[3\bar{\mu}\cdot\bar{r}\,r - \bar{\mu}r^2\right]$$

recall the field for an electric dipole

$$\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^5}\left[3(\bar{p}\cdot\bar{r})\bar{r} - \bar{p}r^2\right]$$

so we see the field far away from a magnetic dipole has the same structure as the field far away from an electric dipole.

This the the magnetic field that would be seen far away from a bar magnet we can use this to understand why opposite poles of a magnet attract, while like poles repel

$$\mathcal{U} = -\vec{\mu}_1 \cdot \vec{B}_2$$

$$= -\vec{\mu}_1 \cdot \frac{\mu_0}{4\pi}\frac{1}{r^5}\left(3\,\vec{\mu}_2\cdot\vec{r}\,\vec{r} - \vec{\mu}_2 r^2\right)$$

if these are in the same direction and $r = (0, 0, z)$

$$\mathcal{U}(z) = \frac{\mu_0}{4\pi}\frac{1}{z^5} \times \mu_1\mu_2\left[-2z^2\right]$$

so the potential energy decreases as the dipoles approach each other, $\sim \frac{1}{z^3}$

similarly if the dipoles have the opposite orientation then the potential energy increases as $z \to 0$.

Analog (- +) (- +) attraction strongest for close charges

Amperes Law

Recall Coulombs law and Gauss Law where equivalent statements of the same thing.

there is a similar relation with the Biot Savart Law and Amperes law.

recall for the long straight wire

$$\bar{B} = \frac{\mu_0 I}{2\pi} \frac{\hat{\theta}}{r}$$

consider

$$\int_0^{2\pi} r\, \hat{\theta} \cdot \bar{B}\, d\theta = \frac{\mu_0 I}{2\pi} \cdot \int_0 d\theta = \mu_0 I$$

here we are integrating along the field line in a circle

$$\boxed{\oint \bar{B} \cdot d\bar{l} = \mu_0 I}$$

In this case the result of this integral is μ_0 x current through the loop.

This result is actually general - It applies to any closed path, and is mathematically equivalent to the Biot Savart law. - It is called Amperes Law

$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I$$

while it is a general result, like gauss law, it is most useful when the integral can be computed using symmetry considerations