Lecture 21

Summary

* magnetic dipoles = current loops

$\bar{\mu} = I A \hat{n}$

$U = -\bar{\mu} \cdot \bar{B}$ potential energy

$\bar{\tau} = \bar{\mu} \times \bar{B}$ torque

* currents are sources for magnetic fields

$$d\bar{B}(\bar{r}) = \frac{\mu_0}{4\pi} \frac{I\, d\bar{\ell}' \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} \quad \text{Biot-Savart}$$

* long straight wire along z-axis

$$\bar{B} = \frac{\mu_0 I}{2\pi} \frac{\hat{\theta}}{r} = \frac{\mu_0 I}{2\pi} \frac{x\hat{y} - y\hat{x}}{r^2} \qquad r = \sqrt{x^2+y^2}$$

. field at origin of circle of radius r

$$\bar{B} = \frac{\mu_0 I}{2r} \hat{z} \qquad \text{(current ccw in xy plane)}$$

. $\bar{F} = I\, d\bar{\ell} \times \bar{B}$, $q\bar{v} \times \bar{B}$

Next we compute the magnetic field due to a magnetic dipole

For simplicity we take as the dipole a circular current loop of radius R in the xy plane with counterclockwise current. As in the case of electric dipoles we are interested in the magnetic field far away from the dipole $|\bar{r}| >> R$

The exact expression for the magnetic field is

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\bar{l}' \times (\bar{r} - \bar{r}')}{|r - r'|^3}$$

where

$$\bar{r}' = (R\cos\theta\,\hat{x} + R\sin\theta\,\hat{y})$$

$$d\bar{l}' = d\bar{r}' = \frac{d\bar{r}'}{d\theta} d\theta = (-R\sin\theta\,\hat{x} + R\cos\theta\,\hat{y})\, d\theta$$

$$\bar{r} = (x\hat{x} + y\hat{y} + z\hat{z})$$

$$|\bar{r} - \bar{r}'|^2 = r^2 + R^2 - 2\,\bar{r}\cdot\bar{r}'$$

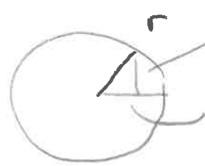
$$= r^2 + R^2 - 2R\,(x\cos\theta + y\sin\theta)$$

$$\bar{r} - \bar{r}' = (x - R\cos\theta)\hat{x} + (y - R\sin\theta)\hat{y} + z\hat{z}$$

We want to look at the leading non 0 power of $\frac{R}{r}$ (small) before we put everything in the integral

① $|r-r'|^2 = r^2\left(1 - \frac{2R}{r}\left(\frac{x}{r}\cos\theta + \frac{y}{r}\sin\theta\right) + \frac{R^2}{r^2}\right)$

note $\frac{x}{r}, \frac{y}{r}$ are



$(\theta' \neq \theta)$

we discard the $\frac{R^2}{r^2}$ term, which is much smaller than the $\frac{R}{r}$ term

let $\Delta = -\frac{2R}{r}\left(\frac{x}{r}\cos\theta + \frac{y}{r}\sin\theta\right)$

with this approximation

$$\frac{1}{|\vec{r}-\vec{r}'|^3} = \frac{1}{r^3(1+\Delta)^{3/2}}$$

let $f(\Delta) = (1+\Delta)^{-3/2}$. For small Δ

$$f(\Delta) = f(0) + \frac{df}{d\Delta}(0)\,\Delta + \underbrace{\frac{1}{2!}\frac{d^2f}{d\Delta^2}(0)\,\Delta^2 + \cdots}$$

ignore these corrections $\sim \left(\frac{R}{r}\right)^2$

$f(0) = 1$

$$\frac{df}{d\Delta}(\Delta) = -\frac{3}{2}(1+\Delta)^{-5/2}$$

$$\frac{df}{d\Delta}(0) = -\frac{3}{2}(1+0)^{-5/2} = -\frac{3}{2}$$

putting these together

$$f(\Delta) \simeq f(0) + \frac{df}{d\Delta}(0)\Delta$$

$$= 1 - \frac{3}{2}\Delta$$

this means

$$\frac{1}{|\bar{r}-\bar{r}'|^3} \simeq \frac{1}{r^3}\left(1-\frac{3}{2}\Delta\right) =$$

$$\frac{1}{r^3}\left(1-\frac{3}{2}\left(-\frac{2R}{r}\right)\left(\frac{x}{r}\cos\theta+\frac{y}{r}\sin\theta\right)\right)$$

$$\frac{1}{r^3}\left(1+\frac{3R}{r}\left(\frac{x}{r}\cos\theta+\frac{y}{r}\sin\theta\right)\right)$$

we also need the cross product

$$d\bar{l}' \times (\bar{r}-\bar{r}') =$$

$$(-R\sin\theta\,\hat{x} + R\cos\theta\,\hat{y})\,d\theta \times \left[(x-R\cos\theta)\hat{x} + (y-R\sin\theta)\hat{y} + z\hat{z}\right] =$$

$$\Big[-R\sin\theta\,(y-R\sin\theta)\hat{z} - R\sin\theta \cdot z\,(-\hat{y})$$

$$R\cos\theta\,(x-R\cos\theta)(-\hat{z}) + R\cos\theta\, z\,(\hat{x})\Big]\,d\theta =$$

$$(-Ry\sin\theta + R^2\sin^2\theta - Rx\cos\theta + R^2\cos^2\theta)\,\hat{z}\,d\theta$$

$$+(Rz\sin\theta\,\hat{y} + Rz\cos\theta\,\hat{x})\,d\theta$$

the next step is to put all of this together and integrate from $0 \to 2\pi$.

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\theta \times \Big[\big(-Ry\sin\theta + R^2\sin^2\theta - Rx\cos\theta + R^2\cos^2\theta \big)\hat{z} + \big(Rz\sin\theta\,\hat{y} + Rz\cos\theta\,\hat{x} \big) \Big] \times$$

$$\times \frac{1}{r^3} \left(1 + \frac{3R}{r} \left(\frac{x}{r}\cos\theta + \frac{y}{r}\sin\theta \right) \right)$$

This looks messy - but all of the integrals are over products of $\sin\theta$'s and $\cos\theta$ terms. The types of integrals are

$$\int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$\int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

$$\int_0^{2\pi} \sin\theta\cos\theta \, d\theta = \int_0^{2\pi} \frac{1}{2}\sin(2\theta) = 0$$

$$\int_0^{2\pi} \sin^3\theta \, d\theta = 0$$

$$\int_0^{2\pi} \cos^3\theta \, d\theta = 0$$

$$\int_0^{2\pi} \sin\theta \cos^2\theta = 0$$

$$\int_0^{2\pi} \cos\theta \sin^2\theta = 0$$

the first second and fifth integrals are obviously 0.

by changing the integration from $0 \to 2\pi$ to $-\pi \to \pi$ the 6 and 8 th integrals are odd functions so the integrals vanish

$$\int_{-\pi}^{0} \sin^3\theta = \int_0^{\pi} \sin^3\theta \qquad \theta = -\theta \text{ in first}$$

$$\int_{\pi}^{0} (-\sin^3\theta)(-d\theta) + \int_0^{\pi} \sin^3\theta \, d\theta =$$

$$-\int_0^{\pi} \sin^3\theta + \int_0^{\pi} \sin^3\theta \, d\theta = 0$$

since $\cos\theta = \sin(\theta + \frac{\pi}{2}) = \sin\theta'$

$$\int_0^{2\pi} \cos^3\theta = \int_0^{2\pi} \sin^3(\tfrac{\pi}{2} - \theta)\, d\theta \qquad \theta' - \tfrac{\pi}{2} - \theta$$

$$-\int_{\frac{\pi}{2}}^{\frac{\pi}{2} - 2\pi} \sin^3\theta' \, d\theta' = \int_{-\pi}^{\pi} \sin^3\theta \, d\theta = 0$$

This means we only have to inspect the integral and locate the term multiplied by $\sin^2\theta \cos^2\theta$ and replace them by π

$$\bar{B} = \frac{\mu_0 I}{4\pi}\left[\left(\frac{R^2\pi + R^2\pi}{r^3} + \frac{3R}{r^4}\left(-\frac{Ry^2}{r}\pi - \frac{Rx^2}{r}\pi\right)\right)\hat{z}\right.$$

$$\left.\frac{3R}{r^4}\left(\frac{RZx}{r}\hat{x}\pi + \frac{RZy}{r}\hat{y}\pi\right)\right]$$

factor out πR^2

$$= \frac{\mu_0}{4\pi}(I\pi R^2)\frac{1}{r^3}\left[\left(2 - \frac{3}{r^2}y^2 - \frac{3}{r^2}x^2\right)\hat{z}\right.$$

$$\left. + \frac{3}{r^2}xz\hat{x} + \frac{3yz}{r^2}\hat{y}\right)$$

we express $2 = 2\frac{x^2+y^2+z^2}{r^2}$

$$= \frac{\mu_0}{4\pi}\mu\frac{1}{r^5}\left(\left(-x^2-y^2-z^2\right)\hat{z} + 3\left(z^2\hat{z} + xz\hat{x} + yz\hat{y}\right)\right)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^5}\left(-r^2\bar{\mu} + 3\,\bar{\mu}\cdot\bar{r}\,\bar{r}\right)$$

$$\bar{B} = \frac{\mu_0}{4\pi}\frac{1}{r^5}\left(3(\bar{\mu}\cdot\bar{r})\bar{r} - r^2\bar{\mu}\right)$$

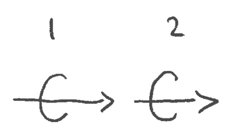
If we compare this to the field for an electric dipole we get

$$\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^5}\left(3(\bar{p}\cdot\bar{r})\bar{r} - r^2\bar{p}\right)$$

which shows that in spite of the difference between the Biot-Savart and Coulomb's law, both lead to the same type of expression for

(1) The field due to a dipole
(2) The potential energy of a dipole
(3) The torque on a dipole

This explains the reason why unlike magnetic poles attract



$$\mathcal{U}_{12} = -\bar{\mu}_1 \cdot \bar{B}_2$$

$$= -I_1 A_1 \hat{z} \cdot \frac{\mu_0}{4\pi} \left(3(\mu_2 z \cdot z) - z^2 \mu_2\right) \hat{z} \frac{1}{z^5}$$

$$= -\mu_1 \mu_2 \frac{\mu_0}{4\pi} \, 2 \, \frac{1}{z^3}$$

which shows that the potential energy decreases as $z \to 0$, which means the poles will attract.

If the dipoles are aligned in the opposite direction then the sign of one magnetic moment changes and the potential will increase as $z \to 0$ $\Rightarrow$ the poles will repel.

This also shows that if we have a bunch of dipoles, the lowest energy configuration is when they all line up – In general this will be disrupted by thermal effects

we can also understand what is happening with electric dipole

$$\ominus \text{———} \oplus \; \ominus \text{———} \oplus$$
$$1 \qquad\qquad 2 \;\; 3 \qquad\qquad 4$$

1 – 3 repel, 2 – 4 repel 2 – 3 attract.
because 2 – 3 are closer their attraction dominate the repulsion of the other 2 pairs (recall the denominator $\to 0$ as the distance $\to 0$)

Amperes law

Consider the field due to the long straight wire

$$\bar{B} = \frac{\mu_0 I}{2\pi}\frac{\bar{\theta}}{r} = \frac{\mu_0 I}{2\pi}\frac{x\hat{y} - y\hat{x}}{r^2}$$

If we integrate this around a loop of radius r

$$\oint \bar{B}\cdot d\bar{l}$$

$$\bar{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$$

$$d\bar{r} = (-r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y})\,d\theta = -(y\,\hat{x} + x\,\hat{y})\,d\theta$$

$$\int_0^{2\pi} \frac{\mu_0 I}{2\pi}\frac{x\hat{y} - y\hat{x}}{r^2}\cdot\left(-\frac{y\hat{x} + x\hat{y}}{r^2}\right) d\theta$$

$$\int_0^{2\pi} \frac{\mu_0 I}{2\pi}\frac{x^2+y^2}{r^2}\,d\theta = \mu_0 I$$

This gives the result that the integral around the closed loop

$$\boxed{\oint \bar{B}\cdot d\bar{l} = \mu_0 I}$$

This result was for the special case of a circular loop around an infinite wire

This result is actually general.
It is called Amperes law.
It applies to any closed loop, and it is exactly equivalent to the Biot savart law

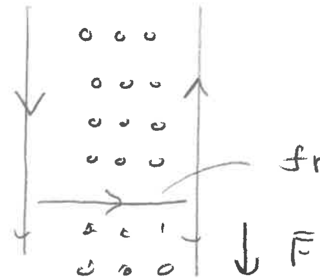
we can also write this as

$$\boxed{\oint_{o} B \cdot d\ell = \mu_0 I = \mu_0 \int_{S} \bar{J} \cdot \hat{n}\, dA}$$

where the integral on the left is a line integral over a closed loop and term on the right is an integral over the surface area enclosed by the loop

Application

① rail gun

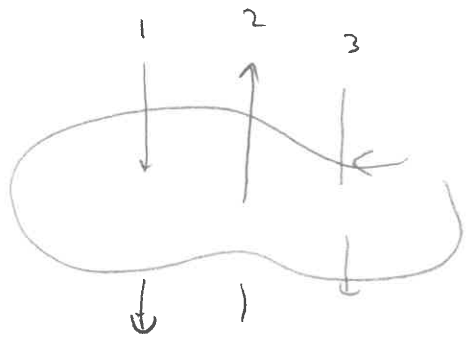


In this case the right hand rule has the magnetic field pointing out of the plane of the paper

This leads to a downward force that accelerates the conductor.

In the book the current is assumed to be so large that it vaporizes the metal and makes a conducting gas that accelerates a projectile.

Amperes Law, superposition



$$\oint \bar{B} \, d\bar{l} = \mu_0 (I_2 - I_3 - I_1)$$

note that the currents in the

same direction as the normal are added, the ones opposite the normal direction are subtracted

Consider multiple loops

By the Biot Savart Law

$$\bar{B} = \frac{\mu_0 I}{2\pi}\frac{\hat{\theta}}{r}$$

$$\oint_{2\,loops} \bar{B}\cdot d\bar{l} = \oint_{loop\,1} \bar{B}\cdot d\bar{l} + \oint_{loop\,2} B\cdot dl = 2\mu_0 I$$

For N loops

$$\oint B\cdot dl = \mu_0 I N$$

Applications

① Long straight wire

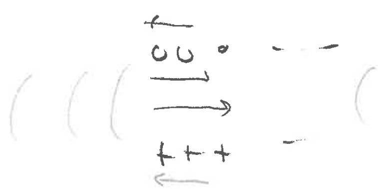
symmetry implies $|\bar{B}|$ only depends on $\perp$ distance from wire the direction is given by the right hand rule. The magnitude is

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B = \mu_0 I$$

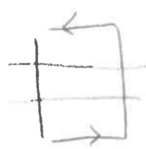
$$\bar{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\theta}$$

which is the same result we got from the Biot Savart law

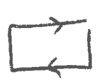
Infinite solenoid



Consider 2 loops



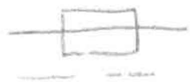
$B_u l - B_L l = \mu_0 \times \text{net current} = 0$



These loops show that the field outside is independent of the distance from the solenoid

If we think of the upper and lower parts as sheets of currents, the field outside of the solenoid should cancel

If we consider



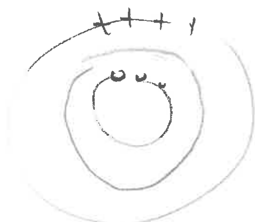
$$\int \bar{B} \cdot dl = Bl = \mu_0 I N$$

$$\boxed{B = \mu_0 I \frac{N}{L}}$$

This gives the strength of the field inside of an infinitely long solenoid

$$\frac{N}{L} = n = \frac{\#\ \text{turns}}{\text{length}}$$

<u>Toroid</u>



$$2\pi r B = \mu_0 I N$$

$$B = \frac{\mu_0 I N}{2\pi r}$$