Lecture 22

Summary

$\bar{F} = q\,\bar{v} \times \bar{B}$

$\bar{F} = I\,d\bar{l} \times \bar{B}$ $\quad d\bar{l}$ direction of current

$\bar{\mu} = IA\,\hat{n}$ $\quad$ magnetic moment

$U = -\bar{\mu} \cdot \bar{B}$

$\bar{\tau} = \bar{\mu} \times \bar{B}$

$d\bar{B} = \dfrac{\mu_0 I}{4\pi} \dfrac{d\bar{l}' \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$ (Biot Savart)

3 applications

① long straight wire (z direction)

$\bar{B} = \dfrac{\mu_0 I}{2\pi r} \dfrac{x\hat{y} - y\hat{x}}{r} = \dfrac{\mu_0 I}{2\pi r}\,\vec{\theta}$

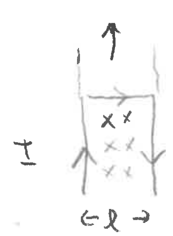
② center of loop

$\bar{B} = \dfrac{\mu_0 I}{2r}\,\hat{z}$

③ magnetic dipole

$B = \dfrac{\mu_0}{4\pi r^5}\left(3\bar{\mu}\cdot\bar{r}\,\bar{r} - r^2\bar{\mu}\right)$

Application: rail gun



$\bar{F} = I\,\bar{l} \times \bar{B}$

multiple loops – use the superposition principle

$$\bar{B} = \int_0^{2\pi} + \int_0^{2\pi} + \int_0^{2\pi} \cdots \frac{\mu_0 I}{4\pi} \frac{d\bar{l}' \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}$$

field at center of multiple loops

$$\bar{B} = N \frac{\mu_0 I}{2r} \hat{z}$$

consider the example of the long straight wire

$$\bar{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}$$

Note the integral

$$\oint_0 \bar{B} \cdot d\bar{l} = \frac{\mu_0 I}{2\pi} \int \frac{x\hat{y} - y\hat{x}}{r^2} \cdot (-y\hat{x} + x\hat{y})\, d\theta$$

$$\bar{r} = (r\cos\theta\, \hat{x} + r\sin\theta\, \hat{y}) = x\hat{x} + y\hat{y}$$

$$d\bar{l} = (-r\sin\theta\, \hat{x} + r\cos\theta\, \hat{y}) = (-y\hat{x} + x\hat{y})\, d\theta$$

$$= \frac{\mu_0 I}{2\pi} \int \frac{d\theta}{r^2} (x^2 + y^2) = \frac{\mu_0 I}{2\pi} \int \frac{r^2}{r^2}\, d\theta = \mu_0 I$$

$$\therefore \quad \boxed{\oint \bar{B} \cdot d\bar{l} = \mu_0 I}$$

At this point this is special to the long straight wire

By symmetry the magnetic field at a given radius has a constant magnitude, with a direction along the circle. This means that

$$\oint \bar{B} \cdot d\bar{l} = 2\pi r B$$

given this integral is $\mu_0 I$, we get

$$|B| = \frac{\mu_0 I}{2\pi r}$$

It turns out that the expression

$$\boxed{\oint \bar{B} \cdot dl = \mu_0 I}$$

where I is the current through the closed curve is a general result, called amperes law

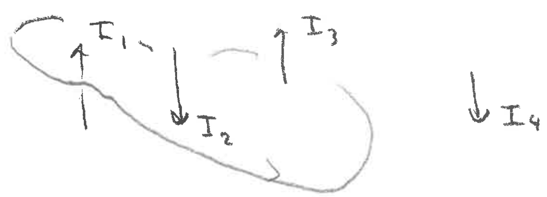
We can express the total current through the curve in terms of the current density

$$\boxed{\oint \bar{B} \cdot d\bar{l} = \mu_0 I = \mu_0 \int_A \bar{J} \cdot \hat{n} \, dA}$$

where the integral on the right is over the area enclosed by the closed curve

This result is called Ampere's law.

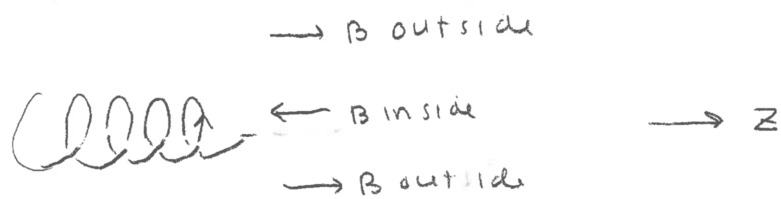
Ampere's law is equivalent to the Biot Savart law. To use it normally requires symmetry. Its important that this is true for any curve



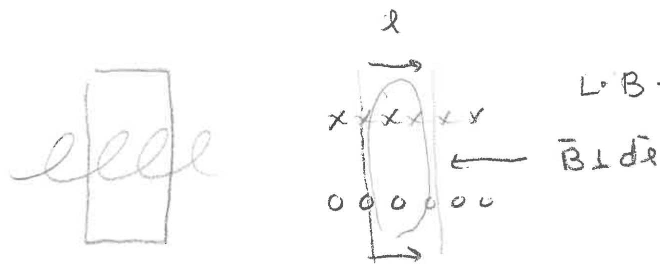
$\oint \bar{B}\cdot d\bar{l} = \mu_0 (I_1 + I_3 - I_2)$

This illustrates that the positive direction of the current is given by the right hand rule

application - infinite solenoid



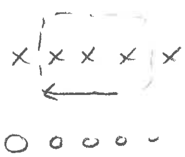
By symmetry this solenoid is invariant with respect to translations along the z axis and rotations about the z axis



$$L\cdot B - L\cdot B = \mu_0\left(\frac{N}{L}L - \frac{N}{L}L\right) = 0$$

so the field outside of an infinite solenoid vanishs by amperes law

Next consider the field inside the solenoid



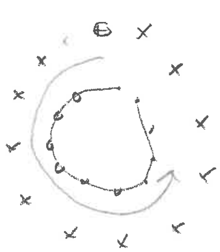
$$Bl = \mu_0 I n l$$

n = Number of turns per unit length

$$\boxed{\bar{B} = \mu_0 I n}$$

the field is constant and independent of the distance from the axis of symmetry

A toroid is a solenoid bent into a circle



using amperes law

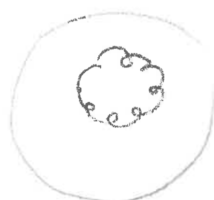
Inside

$$2\pi r B = \mu_0 I N$$

$$B = \frac{\mu_0 I N}{2\pi r}$$

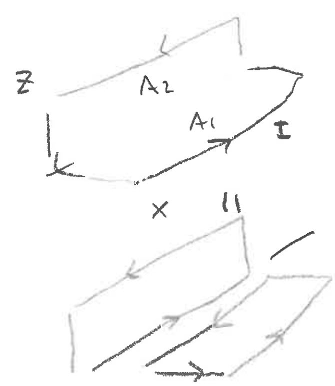
where the direction of the field is given by the right hand rule

outside, the net current in the inside of the curve is 0. Combining this with symmetry gives no field on the outside



Two problems

what is the dipole moment



net current on common edge is 0

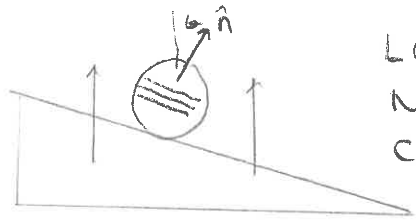
this looks like 2 dipole

$$\bar{\mu} = I A_1 \hat{z} + I A_2 \hat{x}$$

$$= I(A_1 \hat{z} + A_2 \hat{x})$$

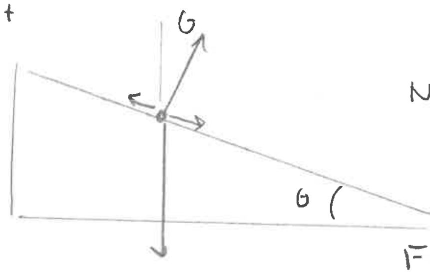
Illustrates how superposition works

example 2 loops



Log of radius R
N loops
current through each loop I

Does not slide - friction balances force down plane



Normal force
$N = mg\cos\theta$

Friction
$F = mg\sin\theta$

torque due to friction (about axis of symmetry)

$Rmg\sin\theta$

direction into plane of paper

magnetic moment NIA in normal direction

$\bar{\tau} = \bar{\mu} \times \bar{B} = NIAB\sin\theta$

out of plane of paper

To keep the log from rolling

$$I = \frac{Rmg\sin\theta}{NAB\sin\theta} = \frac{Rmg}{NAB}$$

Note the result is independent of angle

Req - $\bar{\mu}_{N\,LOOPS} = N\mu_{1\,LOOP}$

So far we have talked about two laws for field

$$\bar{E} = kq \frac{\bar{r}-\bar{r}_q}{|r-r_q|^3} \qquad \oint \bar{E}\cdot\hat{n}\, dA = Q/\epsilon_0$$

$$d\bar{B} = \frac{\mu_0 I}{4\pi} \frac{d\bar{\ell}' \times (\bar{r}-\bar{r}_I)}{|r-r_I|^3} \qquad \oint B\cdot d\bar{\ell} = \mu_0 I$$

The laws on the right and left are equivalent. These are two of the 4 maxwell equations that govern properties of these fields

Next we introduce the third of the maxwell equations. The law involves magnetic flux, which defined analogously to electric flux

$$\Phi_E = \int \bar{E}\cdot\hat{n}\, dA$$

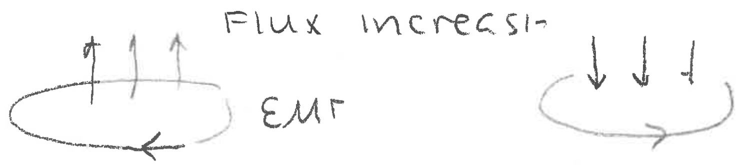
$$\Phi_B = \int \bar{B}\cdot\hat{n}\, dA$$

Faradays law relates the change in magnetic flux / time to an EMF around the boundary of the the area.

Faradays Law

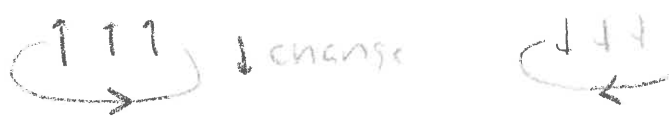
$$\boxed{\frac{d\Phi_B}{dt} = -\mathcal{E}}$$

the − sign indicates that the induced EMF creates a current that opposes the change in the flux



If the field points up but is decreasing then

If the field points down but is decreasing



* it is important that the EMF opposes the change in flux (not necessarily opposing the field)

$$\frac{d}{dt}\int \bar{B}\cdot\hat{n}\, dA = -\mathcal{E}$$

Faraday's law can be expressed in terms of the fields

$$dW = qE \cdot dl$$

integrating around the loop

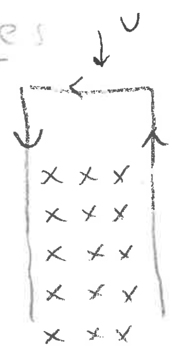
$$\oint W = qV = \int q\bar{E} \cdot d\bar{l}$$

V is the EMF around the loop

$$\boxed{\int_A \bar{B} \cdot \hat{n}\, dA = -\oint E \cdot d\bar{l}}$$

boundary of A.

examples

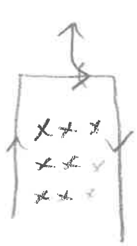


If the wire is moving downward the flux is increasing

$$\Phi = BWVt$$

$$\frac{d\Phi}{dt} = BWV = -\mathcal{E}$$

the induced current reduces the field strength



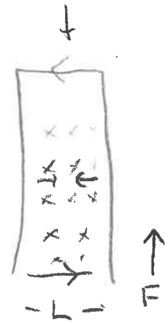
If the wire is moving up the flux is decreasing

$$\Phi = BWVt = -\mathcal{E}$$

In this case the field strength is increased because the flux is being reduced.

The observation that the current creats a field that opposes the change in flux is called Lenz' law

It is useful to look at this from the perspective of work



$$\bar{F} = I\,dl \times B$$

We see pushing down there is an opposing force

$$\bar{F} = I\,dl\,B = ILB$$

$I = \mathcal{E}/R$ R = resistence of wire

$$F = \frac{\mathcal{E}}{R} LB$$

$$= \frac{(BLv)LB}{R}$$

$$= \frac{L^2B^2v}{R}$$

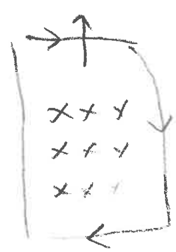
$$\boxed{\text{power} \quad F \cdot \bar{v} = \frac{L^2B^2v^2}{R}}$$

work/time needed to maintain the velocity the power dissipated in the resism is

$$P = I^2 R = \frac{\mathcal{E}^2}{R} = \frac{(BwV)^2}{R}$$

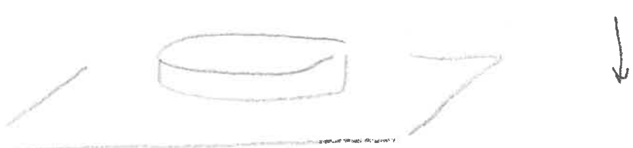
which the same quantity - the work done in pushing the wire into the field is dissipated as heat in the resistor

the same thing happens if we pull the wire out



↓ Fnet down

the magnitudes are all the same



Flux increase through metal conduct

these create eddy currents that cause forces that oppose the change in flux

Differential form of Faraday Law

consider a small square of size $\Delta x \, \Delta y$

$$\frac{\partial \Phi}{\partial t} \simeq \frac{\partial}{\partial t}\left(B_z(x\,y\,z\,t)\,\Delta x\,\Delta y \right)$$

$$= \frac{\partial B_z}{\partial t}(x\,y\,z\,t)\,\Delta x\,\Delta y$$

$$= -\int \bar{E}\cdot d\ell \approx$$

$$= -\Big(E_x(x\,y\,z\,t)\,\Delta x + E_y(x+\Delta x, y\,z\,t)\,\Delta y - E_x(x, y+\Delta y\,t)\,\Delta x - E_y(x, y\,z\,t)\,\Delta y \Big)$$

$$= -\Delta x\left(E_x(x\,y\,z\,t) - E_x(x, y+\Delta y, z\,t) \right) - \Delta y\left(E_y(x+\Delta x\,y\,z\,t) - E_y(x\,y\,z\,t) \right)$$

$$E_x(x, y+\Delta y, z\,t) - E(x\,y\,z\,t) \simeq \frac{\partial E_x}{\partial y}(x\,y\,z\,t)\,\Delta y$$

$$E_y(x+\Delta x, y\,z\,t) - E_y(x, y\,z\,t) \simeq \frac{\partial E_y}{\partial x}(x\,y\,z\,t)\,\Delta y$$

using these in the above

$$= \Delta x\,\Delta y\,\frac{\partial E_x}{\partial y}(x\,y\,z\,t) - \Delta x\,\Delta x\,\frac{\partial E_y}{\partial x}(x\,y\,z\,t)$$

putting everything together

$$\left(\frac{\partial B_z}{\partial t}(xyzt) - \frac{\partial E_x}{\partial y}(xyzt) + \frac{\partial E_y}{\partial x}(xyzt) \right) \Delta x \Delta y = 0$$

this is approximate, but it becomes exact in the limit that $\Delta x \Delta y \to 0$

If we consider the yz zx planes we will get similar equat

$$\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

$$\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0$$

these equations can be expressed vector form

$$\boxed{\frac{\partial \bar{B}}{\partial t} + \vec{\nabla} \times \bar{E} = 0}$$

we get one more interesting identity - if we integrate over $\int \cdot \hat{n} \, dA$

$$\int_A \left(\frac{\partial B}{\partial t} \cdot n \, dA + (\nabla \times E) \cdot \hat{n} \, dA \right) = 0$$

$$\frac{d\Phi}{dt} = -\oint \bar{E} \cdot d\bar{l}$$

This gives

$$\boxed{\oint \bar{E}\cdot d\bar{\ell} = \int (\bar{\nabla}\times\bar{E})\cdot\hat{n}\, dA}$$

while we derived this using Faraday's law, the field is an arbitrary vector field. This is a general result for any vector field called Stokes theorem where the integrals are over the area surrounded by the curve

recall

$$\oint \bar{B}\cdot d\bar{\ell} = \mu_0 \int \bar{J}\cdot\hat{n}\, dA$$

$$\shortparallel$$

$$\int (\nabla\times B)\cdot \hat{n}\, dA$$

$$\int (\nabla\times B - \mu_0 J)\cdot\hat{n}\, dA = 0$$

$$\boxed{\nabla\times\bar{B} = \mu_0 \bar{J}}$$

differential form of Biot Savart Law