Lecture 24

Faraday's Law

$$\Phi = \int \bar{B}\cdot\hat{n}\, dA = \text{magnetic flux}$$

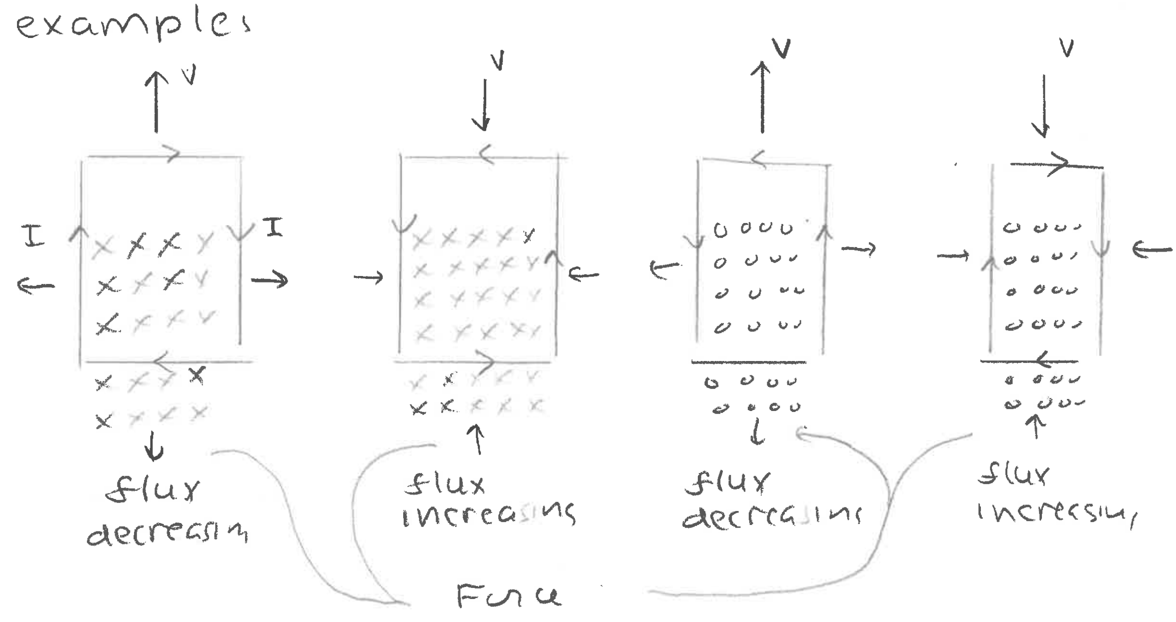
$$\frac{d\Phi}{dt} = -\oint \bar{E}\cdot d\bar{\ell}$$

* The integral is around the boundary of the area, counter clockwise relative to the normal vector for the area.

* The − sign means that the created EMF opposes the <u>change</u> in the magnetic flux

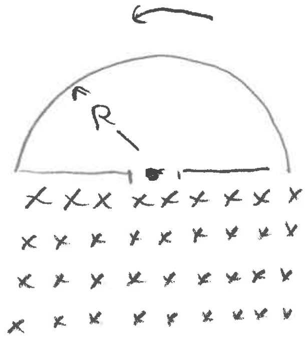
This is called Lenz's law

4 examples



← illustrates − sign →

square wave generator



$\phi = \omega t$

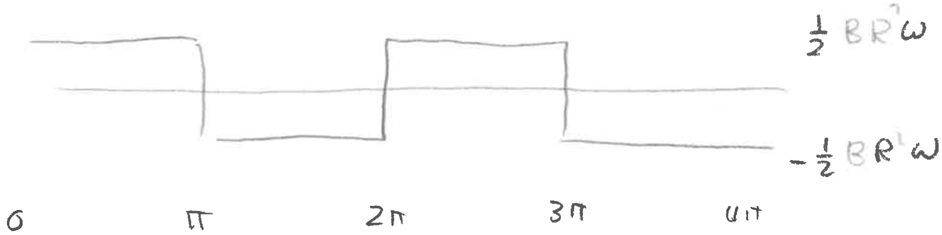
$\theta: 0 \to \pi$ flux increases

$$\Phi = B \cdot R^2 \frac{\theta}{2} = BR^2 \frac{\omega t}{2}$$

$\theta: \pi - 2\pi$ flux decreases

$$\Phi = -BR^2 \frac{\theta}{2} = -BR^2 \frac{\omega t}{2}$$

$$\frac{d\Phi}{dt} = \frac{1}{2} BR^2\omega \begin{cases} + & 2n\pi < \theta < (2n+1)\pi \\ - & (2n+1)\pi < \theta < (2n+2)\pi \end{cases}$$



periodically flipping the wires at each multiple of π $_{+}|\!\bigcirc\!|_{-}$ give a dc emf.

differential form of Faradays Law

Let Δx and Δy be small and consider

$$\frac{1}{\Delta x \Delta y} \int_x^{x+\Delta x} dx' \int_y^{y+\Delta y} dy' \frac{\partial \bar{B}}{\partial t}(x' y' z t) \cdot \hat{z} =$$

$$- \frac{1}{\Delta x \Delta y} \Big(\int_x^{x+\Delta x} E_x(x' y z t)\, dx' + \int_y^{y+\Delta y} E_y(x+\Delta x, y' z t)\, dy' +$$

$$\int_{x+\Delta x}^{x} dx' E_x(x', y+\Delta y, z t) + \int_{y+\Delta y}^{y} E(x, y' z t)\, dy' \Big)$$

The differential form of Faradays law follows by taking the limit as $\Delta x \; \Delta y \to 0$

To calculate this we only need to retain the leading terms

$$\frac{\Delta x \Delta y}{\Delta x \Delta y} \frac{\partial B_z}{\partial t}(x y z t) + \frac{O(\Delta^3)}{\Delta x \Delta y} =$$

$$- \frac{1}{\Delta x \Delta y} \Big(E_x(xyzt)\Delta x + E_y(x+\Delta x, y, z\, t)\Delta y$$

$$- E_x(x\; y+\Delta y\; z\, t)\Delta x - E_y(x y z t)\Delta y$$

$$O(\Delta^3) \Big) =$$

$$- \frac{1}{\Delta x \Delta y} \Big(\frac{\partial E_y}{\partial x}(x y z t)\Delta x \Delta y - \frac{\partial E_x}{\partial y}(x y z t)\Delta y \Delta x$$

$$+ O(\Delta^3) \Big)$$

In the limit $\Delta x \; \Delta y \to 0$ what survives

$$\boxed{\frac{\partial B_z}{\partial t}(x y z t) = - \frac{\partial E_y}{\partial x}(x y z t) + \frac{\partial E_x}{\partial y}(x y z t)}$$

using the yz and zx planes gives

$$\boxed{\begin{aligned} \frac{\partial B_x}{\partial t}(x y z t) &= - \frac{\partial E_z}{\partial y}(x y z t) + \frac{\partial E_y}{\partial z}(x y z t) \\ \frac{\partial B_y}{\partial t}(x y z t) &= - \frac{\partial E_x}{\partial z}(x y z t) + \frac{\partial E_z}{\partial x}(x y z t) \end{aligned}}$$

we can express this in terms of unit vectors

$$\hat{z}\,\frac{\partial B_z}{\partial t}(x\,y\,z\,t) = \underbrace{\left(-\hat{x}\frac{\partial}{\partial x} - \hat{y}\frac{\partial}{\partial y}\right)\times\left(\hat{x}E_x + \hat{y}E_y\right)}$$

the cross products give $\pm\hat{z}$

we can summarize all 3 equations using

$$\frac{\partial \bar{B}}{\partial t}(x\,y\,z\,t) = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\times\bar{E}$$

recall

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

which allows us to write Faradays law as

$$\boxed{\frac{\partial \bar{B}}{\partial t} = -\bar{\nabla}\times\bar{E}}$$

If we integrate $\int \hat{n}\,dA\,\cdot$ $\Rightarrow$

$$\frac{d}{dt}\int \bar{B}\cdot\hat{n}\,dA = -\int(\bar{\nabla}\times\bar{E})\cdot\hat{n}\,dA$$

using faradays law

$$\frac{d}{dt}\int \bar{B}\cdot\hat{n}\,dA = -\oint \bar{E}\cdot d\bar{\ell}$$

equating both sides gives

$$* \quad \boxed{\int_A (\bar{\nabla} \times \bar{E}) \cdot \hat{n}\, dA = \oint \bar{E} \cdot d\bar{l}}$$

Here the magnetic field does not appear. This is a general result for any vector field

We can use this in amperes law to get

$$\oint \bar{B} \cdot dl = \mu_0 I = \mu_0 \int \bar{J} \cdot \hat{n}\, dA$$
$$= \int (\bar{\nabla} \times \bar{B}) \cdot \hat{n}\, dA$$

since this must hold for arbitrary surfaces we get the differential form of Ampere's Law

$$\boxed{\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}}$$

The integral relation that we derived is called stoke's theorem.

Inductors

consider a solenoid with n turns / length.
the magnetic field in the solenoid follows from ampere's law

$$\ell B = \mu_0 n I \ell$$

$$B = \mu_0 n I$$

the total flux is

$$\Phi = B A n \ell$$

putting these together

$$\Phi = (\mu_0 n I) A n \ell$$
$$\Phi = \mu_0 n^2 A \ell \cdot I$$

the ratio of the magnetic flux to the current in the solenoid is called the inductance "L" of the solenoid

$$\boxed{L \equiv \frac{\Phi}{I} = \frac{\text{Tesla} \cdot (\text{meter})^2}{\text{ampere}} = \text{Henry}}$$

In this case

$$\boxed{L = \mu_0 n^2 A L \qquad \text{solenoid}}$$

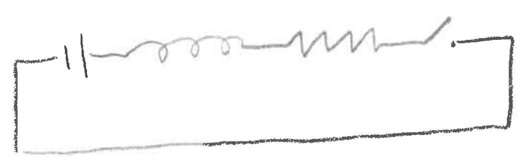
an inductor is a circuit element.
It is expressed by a symbol

that looks like a solenoid. Since

$$\boxed{\frac{d\Phi}{dt} = L\frac{dI}{dt} = -\mathcal{E}}$$

gives the voltage drop across the inductor

consider RL circuits;



Initially the circuit is open.
when the switch is closed the loop equation gives

$$\boxed{\mathcal{E} - L\frac{dI}{dt} - IR = 0}$$

Initially there is no current so $I(0)=0$. This is a differential equation

$$L \frac{dI}{dt} = \mathcal{E} - IR$$

$$\frac{dI}{\mathcal{E} - IR} = \frac{dt}{L}$$

we write this as

$$\frac{dI}{I - \mathcal{E}/R} = -\frac{R}{L}\,dt$$

Let $u = I - \mathcal{E}/R \qquad du = dI$

$$\frac{du}{u} = -\frac{R}{L}\,dt$$

integrating

$$\int_{I(0) - \mathcal{E}/R}^{I(t) - \mathcal{E}/R} \frac{du}{u} = -\int_0^t \frac{R}{L}\,dt$$

the integral gives

$$\ln\left(I(t) - \mathcal{E}/R\right) - \ln\left(I(0) - \mathcal{E}/R\right) = -\frac{R}{L}t$$

$$\ln\left(\frac{I(t) - \mathcal{E}/R}{I(0) - \mathcal{E}/R}\right) = \ln\left(\frac{\mathcal{E}/R - I(t)}{\mathcal{E}/R - I(0)}\right) = -\frac{R}{L}t$$

taking exponentials of both sides

$$\frac{\mathcal{E}/R - I(t)}{\mathcal{E}/R - I(0)} = e^{-\frac{R}{L}t}$$

solving for $I(t)$

$$\mathcal{E}/R - I(t) = (\mathcal{E}/R - I(0))\, e^{-\frac{R}{L}t}$$

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right) + I(0)\, e^{-\frac{R}{L}t}$$

If $I(0) = 0$ this becomes

$$\boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

at time 0 the current vanishes -
after a long time it becomes $\mathcal{E}/R$

once the full current is established
and the battery is shorted out

$$0 - L\frac{dI}{dt} - IR = 0$$

$$I(0) = \frac{\mathcal{E}}{R}$$

$$\frac{dI}{dt} = -\frac{R}{L}I$$

$$\frac{dI}{I} = -\frac{R}{L}dt$$

integration

$$\int_{\mathcal{E}/R}^{I(t)} \frac{dI}{I} = -\int_0^t \frac{R}{L}t = -\frac{R}{L}t$$

$$\ln I(t) - \ln(\mathcal{E}/R) = -\frac{R}{L}t$$

$$\frac{I(t)}{\mathcal{E}/R} = e^{-\frac{R}{L}t}$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R} e^{-\frac{R}{L}t}}$$

in this case the current eventually goes to 0

the power dissipated in the resistor as a function of time

$$P = I^2(t) R$$

$$= \frac{\mathcal{E}^2}{R^2} R e^{-\frac{2R}{L}t}$$

$$\boxed{P = \frac{\mathcal{E}^2}{R} e^{-\frac{2R}{L}t}}$$

the total energy stored in the inductor is

$$\int_0^\infty P(t)\,dt = \int_0^\infty \frac{\mathcal{E}^2}{R} e^{-\frac{2R}{L}t}\,dt$$

$$u = \frac{2R}{L}t \qquad du = \frac{2R}{L}dt \qquad dt = \frac{L}{2R}du$$

$$\text{energy} = \frac{\mathcal{E}^2}{R} \cdot \frac{L}{2R} \int_0^\infty e^{-u}\,du$$

$$= \frac{\mathcal{E}^2}{2R^2} L \left(-e^{-\infty} + e^0\right)$$

this means that the energy stored in the inductor is

$$\boxed{\text{energy} = \left(\frac{\mathcal{E}}{R}\right)^2 \frac{L}{2} = \frac{1}{2} I(0)^2 L}$$

this shows that inductors store energy. Note

$$\Phi = IL \qquad I = \frac{\Phi}{L}$$

$$\text{energy} = \frac{1}{2}\left(\frac{\Phi^2}{L^2}\right)L$$

$$= \frac{1}{2}\frac{\Phi^2}{L}$$

$$\text{energy} = \frac{1}{2}\frac{(BNA)^2}{L}$$

for the case of a solenoid

$$L = \mu_0 n^2 A \ell$$

$$\text{energy} = \frac{1}{2}\frac{(BNA)^2}{\mu_0 n^2 A \ell} =$$

$$= \frac{1}{2\mu_0}\frac{(BN)^2 A\ell}{(n\ell)^2}$$

$$= \frac{1}{2\mu}\frac{(BN)^2}{N^2}A\ell$$

$$= \frac{1}{2\mu} B^2 A\ell$$

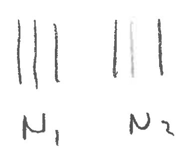
$$\boxed{\frac{\text{energy}}{A\ell} = \frac{1}{2\mu} B^2}$$

since for a solenoid all of the magnetic field is inside of the solenoid.

The energy density in a magnetic field is

$$\boxed{\left(\begin{array}{c}\text{energy}\\ \text{density}\end{array}\right) = \frac{1}{2\mu_0} B^2}$$

Mutual Induction.- consider 2 solenoids adjacent to each other the first with N_1 turns and the second with N_2 turns



The magnetic field in the second inductor due to a current in the first inductor "

$$B_1 = \mu_0 N I$$

The resulting magnetic flux

This is not exact because the solenoid is not infinite and we are at the edges

The flux through the second solenoid is

$$\Phi_2 = N_2 B_1 A_2$$
$$= N_2(\mu_0 N_1 I_1) A_2$$
$$= (\mu_0 N_1 N_2 A) I_1$$

If we differentiate both sides of this equation we get

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -(\mu_0 N_1 N_2 A)\frac{dI_1}{dt}$$

In this idealized example

$$M = \mu_0 N_1 N_2 A$$

is called the mutual inductance,
In order to properly treat the non uniformity of the field we define

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}$$
$$\mathcal{E}_1 = -M_{12}\frac{dI_2}{dt}$$

(changing the current in each coil induces an emf in the other coil.

while it is not immedictely obvious in general

$$M_{12} = M_{21} = M$$

is called the mutul inductance