Lecture 24

Faraday's Law

\[ \Phi = \int \mathbf{B} \cdot \mathbf{n} \, d\mathbf{A} = \text{magnetic flux} \]

\[ \frac{d\Phi}{dt} = -\oint \mathbf{E} \cdot d\mathbf{e} \]

* The integral is around the boundary of the area, counter clockwise relative to the normal vector for the area.

* The minus sign means that the created EMF opposes the change in the magnetic flux.

This is called Lenz's law.

4 examples

\[ \text{ Flux decreases } \quad \text{ Flux increases } \quad \text{ Flux decreases } \quad \text{ Flux increases } \]

\[ \bigg\{ \text{ illustrates - sign } \bigg\} \]
square wave generator

\[ \phi = \omega t \]

0: 0 → π flux increases
\[ \mathcal{E} = B \cdot R^2 \frac{\phi}{2} = BR^2 \frac{\omega t}{2} \]

0: π → 2π flux decreases
\[ \mathcal{E} = -BR^2 \frac{\phi}{2} = -BR^2 \frac{\omega t}{2} \]

\[ \frac{d\phi}{dt} = \frac{1}{2} BR^2 \omega \left\{ \begin{array}{ll}
+ & 2n\pi < \phi < (2n+1)\pi \\
- & (2n+1)\pi < \phi < (2n+2)\pi
\end{array} \right. \]

periodically flipping the wires at each multiple of π give a dc emf.

differential form of Faraday Law

Let \( \Delta x \) and \( \Delta y \) be small and consider:

\[ \frac{1}{\Delta x \Delta y} \int_{x}^{x+\Delta x} dx' \int_{y}^{y+\Delta y} dy' \frac{\partial B}{\partial t} (x', y', z, t) \cdot \hat{n} = \]

\[ = \frac{1}{\Delta x \Delta y} \left( \int_{x}^{x+\Delta x} \mathcal{E}_x (x', y, z, t) \, dx' + \int_{y}^{y+\Delta y} \mathcal{E}_y (x+\Delta x, y, z, t) \, dy' + \int_{x}^{x+\Delta x} \mathcal{E}_x (x', y+\Delta y, z, t) \, dx' + \int_{y}^{y+\Delta y} \mathcal{E} (x, y, z, t) \, dy' \right) \]
the differential form of Faraday's law follows by taking the limit as $\Delta x \Delta y \to 0$.

To calculate this, we only need to retain the leading terms:

$$\frac{\Delta x \Delta y}{\Delta x \Delta y} \frac{\partial B_2}{\partial t}(x y z t) + \frac{O(\Delta^3)}{\Delta x \Delta y} =$$

$$- \frac{1}{\Delta x \Delta y} \left( E_x(x y z t) \Delta x + E_y(x y z t) \Delta x \Delta y - E_x(x y z t) \Delta x - E_y(x y z t) \Delta y \right) \frac{O(\Delta^3)}{\Delta x \Delta y}$$

$$- \frac{1}{\Delta x \Delta y} \left( \frac{\partial E_y}{\partial x}(x y z t) \Delta x \Delta y - \frac{\partial E_x}{\partial y}(x y z t) \Delta x \Delta y + O(\Delta^3) \right)$$

In the limit $\Delta x \Delta y \to 0$, what survives:

$$\frac{\partial B_2}{\partial t}(x y z t) = - \frac{\partial E_y}{\partial x}(x y z t) + \frac{\partial E_x}{\partial y}(x y z t)$$

Using the $yz$ and $xz$ planes gives:

$$\frac{\partial B_x}{\partial t}(x y z t) = - \frac{\partial E_z}{\partial y}(x y z t) + \frac{\partial E_y}{\partial z}(x y z t)$$

$$\frac{\partial B_y}{\partial t}(x y z t) = - \frac{\partial E_z}{\partial x}(x y z t) + \frac{\partial E_x}{\partial z}(x y z t)$$
we can express this in terms of unit vectors
\[
\frac{\partial \mathbf{B}}{\partial t} (x,y,z,t) = \left( -\hat{x} \frac{\partial}{\partial x} - \hat{y} \frac{\partial}{\partial y} - \hat{z} \frac{\partial}{\partial z} \right) \times \left( \hat{x} \mathbf{E}_x + \hat{y} \mathbf{E}_y + \hat{z} \mathbf{E}_z \right)
\]
the cross product gives
we can summarize all these equations using
\[
\frac{\partial \mathbf{B}}{\partial t} (x,y,z,t) = - \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \mathbf{E}
\]
recall
\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\]
which allows us to write Faraday's law as
\[
\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E}
\]
If we integrate \( \int \hat{n} \cdot d\mathbf{A} \), \( \nabla \cdot \mathbf{E} = 0 \)
\[
\frac{d}{dt} \int \mathbf{B} \cdot \hat{n} d\mathbf{A} = - \int (\nabla \times \mathbf{E}) \cdot \hat{n} d\mathbf{A}
\]
using Faraday's law
\[
\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} = - \int \mathbf{E} \cdot d\mathbf{l}
\]
equating both sides gives

\[ \int_A (\nabla \times \vec{E}) \cdot \hat{n} \, dA = \oint \vec{E} \cdot d\vec{l} \]

Here the magnetic field does not appear. This is a general result for any vector field.

We can use this in Ampère's law to get

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \int \vec{J} \cdot \hat{n} \, dA \]

\[ = \int (\nabla \times \vec{B}) \cdot \hat{n} \, dA \]

Since this must hold for arbitrary surfaces, we get the differential form of Ampère's law

\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]

The integral relation that we derived is called Stokes' theorem.
Inductors

Consider a solenoid with n turns/length. The magnetic field in the solenoid follows from amperes' law:

\[ lB = \mu_0 n I l \]

\[ B = \mu_0 n I \]

The total flux is

\[ \Phi = B A n l \]

Putting these together:

\[ \Phi = (\mu_0 n I) A n l \]

\[ \Phi = \mu_0 n^2 A l I \]

The ratio of the magnetic flux to the current in the solenoid is called the inductance "L" of the solenoid:

\[ L = \frac{\Phi}{I} = \frac{\text{Tesla} \cdot (\text{meters})^2}{\text{ampere}} = \text{Henry} \]

In this case:

\[ L = \mu_0 n^2 A l \text{ solenoid} \]
an inductor is a circuit element. It is expressed by a symbol

that looks like a solenoid. Since

\[
\frac{d\Phi}{dt} = L \frac{dI}{dt} = -\mathcal{E}
\]
gives the voltage drop across the inductor.

Consider RL circuit:

Initially the circuit is open, when the switch is closed the loop equation gives

\[
\mathcal{E} - L \frac{dI}{dt} - IR = 0
\]

Initially there is no current so \( I(0) = 0 \), this is a differential equation.
\[ L \frac{dI}{dt} = E - IR \]
\[ \frac{dI}{E - IR} = \frac{dt}{L} \]

we write this as
\[ \frac{dI}{I - \frac{E}{R}} = -\frac{R}{L} dt \]

Let \( u = I - \frac{E}{R} \quad du = dI \)
\[ \frac{du}{u} = -\frac{R}{L} dt \]

integrating
\[ \int_{I(0)}^{I(t)} \frac{du}{u} = -\int_{0}^{t} \frac{R}{L} dt \]

the integral gives
\[ \ln \left( \frac{I(t)}{I(0)} \right) - \ln \left( \frac{E}{R} \right) = -\frac{R}{L} t \]
\[ \ln \left( \frac{I(t)}{E/R - I(0)} \right) = -\frac{R}{L} t \]

Taking exponentials of both sides
\[ \frac{E/R - I(t)}{E/R - I(0)} = e^{-\frac{R}{L} t} \]

solving
\[ I(t) \]
\[ \frac{E}{R} - I(t) = (\frac{E}{R} - I(0)) e^{-\frac{R}{L} t} \]

\[ I(t) = \frac{E}{R} \left(1 - e^{-\frac{R}{L} t}\right) + I(0) e^{-\frac{R}{L} t} \]

If \( I(0) = 0 \), this becomes

\[ I(t) = \frac{E}{R} \left(1 - e^{-\frac{R}{L} t}\right) \]

At time 0, the current vanishes; after a long time it becomes \( \frac{E}{R} \).

Once the full current is established and the battery is shorted out,

\[ 0 = L \frac{dI}{dt} + IR \]

\[ I(0) = \frac{E}{R} \]

\[ \frac{dI}{dt} = -\frac{R}{L} I \]

\[ \int \frac{dI}{I} = -\int_{0}^{t} \frac{R}{L} dt = -\frac{R}{L} t \]

\[ \ln I(t) - \ln \left(\frac{E}{R}\right) = -\frac{R}{L} t \]
\[ I(t) = e^{-\frac{R}{L}t} \]

\[ I(t) = \frac{E}{R} e^{-\frac{R}{L}t} \]

In this case the current eventually goes to 0.

The power dissipated in the resistor as a function of time is

\[ P = I^2(t)R = \frac{E^2}{R^2} e^{-\frac{2R}{L}t} \]

\[ P = \frac{E^2}{R} e^{-\frac{2R}{L}t} \]

The total energy stored in the inductor is

\[ \int_0^\infty P(t)dt = \int_0^\infty \frac{E^2}{R} e^{-\frac{2R}{L}t} dt \]

\[ u = \frac{2R}{L} \quad du = \frac{2R}{L} dt \quad di = \frac{L}{2R} du \]

energy = \[ \frac{E^2}{R} \cdot \frac{L}{2R} \int_0^\infty e^{-u} du \]

\[ = \frac{E^2}{2R^2L} (-e^{-\alpha} + e^{\alpha}) \]
This means that the energy stored in the inductor is

\[
\text{energy} = \left(\frac{\mathcal{E}}{R}\right)^2 \frac{L}{2} = \frac{1}{2} I^2 L
\]

This shows that inductors store energy. Note

\[
\Phi = IL, \quad I = \frac{\Phi}{L}
\]

\[
\text{energy} = \frac{1}{2} \left(\frac{\Phi^2}{L^2}\right) L = \frac{1}{2} \frac{\Phi^2}{L}
\]

\[
\text{energy} = \frac{1}{2} \frac{(BNA)^2}{L}
\]

For the case of a solenoid

\[
L = \mu_0 N^2 A L
\]

\[
\text{energy} = \frac{1}{2} \frac{(BNA)^2}{\mu_0 N^2 A L} = \frac{1}{2 \mu_0} \frac{(B^2 N A L)^2}{(N^2 A L)^2} = \frac{1}{2 \mu_0} \frac{(B^2 N A L)^2}{N^2 A L} = \frac{1}{2 \mu_0} B^2 A L
\]
\[
\frac{\text{energy}}{\text{Ae}} = \frac{1}{2\mu} B^2
\]

Since in a solenoid all of the magnetic field is inside of the solenoid.

The energy density in a magnetic field is

\[
(\text{energy density}) = \frac{1}{2\mu} B^2
\]

**Mutual Induction:** Consider 2 solenoids adjacent to each other, the first with \( N_1 \) turns and the second with \( N_2 \) turns.

The magnetic field in the second inductor due to a current in the first inductor is

\[
B_i = \mu_0 N_1
\]

The resulting magnetic flux
This is not exact because the solenoid is not infinite and we are at the edges.

The flux through the second solenoid is

$$\Phi_2 = N_2 B_i A_i$$

$$= N_2 (\mu_0 N_i) A_i$$

$$= (\mu_0 N_i N_i A) I_i$$

If we differentiate both sides of this equation we get

$$E_2 = -\frac{d\Phi_2}{dt} = -(\mu_0 N_i N_i A) \frac{dI_i}{dt}$$

In this idealized example

$$M = \mu_0 N_i N_i A$$

is called the mutual inductance.

In order to properly treat the non-uniformity of the field, we define

$$E_2 = -M_{21} \frac{dI_1}{dt}$$

$$E_1 = -M_{12} \frac{dI_2}{dt}$$

(changing the current in each coil induces an emf in the other coil.)
while it is not immediately obvious, in general

\[ M_{12} = M_{21} = M \]

called the mutual inductance