Lecture 26

Exam

Average = 102/170

St. Dev = 35

(1a) $P = IV \rightarrow I = \frac{P}{V} = \frac{100W}{100V} = 1A$

(1b) $R = V/I \rightarrow R = \frac{100V}{1A} = 100\Omega$

(1c) $R_T = 10R = 1000\Omega$

(1d) $I = V/R = 100V/1000\Omega = .1A$

(1e) $P = IV = (.1A)(100V) = 10$ watts

(2a) $C_T = \frac{(C_1+C_2)C_3}{C_1+C_2+C_3} = \frac{2}{3}C = \frac{4}{3}\times10^{-6}F$

(2b) $Q = CV = \frac{4}{3}\times10^{-6}\times 12 = 16\times10^{-6}\ C = 1.6\times10^{-5}C$

(2c) $V = \frac{Q}{2C} = (\frac{4}{3}\times10^{-6}\times12)(2\times2\times10^{-6})^{-1} = \frac{12}{3}V = 4V$

(2d) energy $= \frac{1}{2}CV^2 = \frac{1}{2}\times(2\times10^{-6})(12-4)^2 = 64\times10^{-6}J = 6.4\times10^{-5}J$

(3a) Amperes Law + Superposition

$\bar{B} = \frac{\mu_0 I}{2\pi d}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right)\hat{y} = \frac{\mu_0 I}{2\pi d}\cdot\frac{13}{12}\hat{y}$

(3b) $\bar{0}$ fields are equal and opposite

(3c) $\bar{F}/\ell = I\frac{\mu_0 I}{2\pi d}\left(\frac{1}{1}+\frac{1}{2}\right)(-\hat{x}) = -\frac{\mu_0 I^2}{\pi d}\left(\frac{3}{4}\right)$

(3d) $\bar{F} = I d\ell\times B = 0$ for $I = 0$

(4a) $Q(t) = Q(0)e^{-t/RC}$ $\quad Q(0) = CV = (1\times10^{-6})\times(12)\ C.$

$\frac{1}{e} = e^{-t/RC} \Rightarrow -\ln e = -1 = \ln e^{-t/RC} = -t/RC$

$R = \frac{t}{C} = \frac{.1}{1\times10^{-6}} = 10^5\Omega$

(4b) $Q(t) = CVe^{-10t} = (12\times10^{-6})e^{-10t}\ C$

(4c) $I(t) = \frac{dQ}{dt} = \frac{V}{R}e^{-10t} = \frac{12}{10^5}e^{-10t} = 1.2\times10^{-4}e^{-10t}$

(4d) $P = I^2(t)R = V^2/R\ e^{-20t} = 144\times10^{-5}e^{-20t}$ WATTS

Faraday's Law

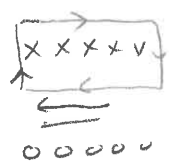
$$\Phi = \int \bar{B} \cdot \hat{n} \, dA$$

$$\frac{d\Phi}{dt} = -EMF = -\oint \bar{E} \cdot d\bar{l} \qquad \frac{\partial \bar{B}}{\partial t} = -\bar{\nabla} \times \bar{E}$$

change in flux creates a field that creates a current that opposes the change in magnetic flux.

consider a solenoid with n turns of wire per unit length

the magnetic field inside the solenoid can be computed using amperes law



$$|\bar{B}| l = \mu_0 I n l \rightarrow B = \mu_0 I n$$

the total flux in length l is

$$\Phi = \pi R^2 n l B$$

$$\boxed{\Phi = \pi R^2 l \cdot \mu_0 n^2 I}$$

In this case we see that the magnetic flux in this solenoid is proportional to the current

We write this as

$$\Phi = LI$$

or

$$\boxed{L = \frac{\Phi}{I}}$$

The quantity L is called the inductance of the solenoid. For our ideal solenoid

$$L_s = \pi R^2 \ell \mu_0 n^2 \qquad \text{(ideal solenoid)}$$

which are all geometric factors

If we differentiate Φ with respect to time we get the relation

$$\frac{d\Phi}{dt} = -\mathcal{E} = L\frac{dI}{dt}$$

or

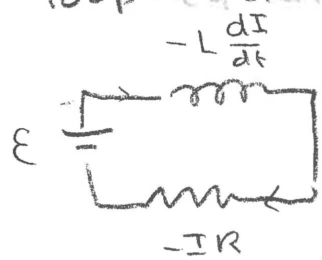
$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

The MKS unit of inductance is

$$1 \text{ Henry} = \frac{1 \text{ Tesla Meter}^2}{\text{coulomb}} \cdot \text{second}$$

Inductors are circuit elements like capacitors, the notation is

The loop equation



To analyze this loop

$$-L\frac{dI}{dt} - IR + \mathcal{E} = 0$$

$$\boxed{\frac{dI}{dt} = \frac{1}{L}(\mathcal{E} - IR)}$$

This is another differential equation (for an RL circuit)

To solve this

$$\frac{\frac{dI}{dt}}{IR - \mathcal{E}} = -\frac{1}{L}$$

multiply both sides by R

$$\frac{\frac{dI}{dt}}{I - \mathcal{E}/R} = -\frac{R}{L}$$

or $$\frac{dI}{I - \mathcal{E}/R} = -\frac{R}{L}\,dt$$

integrating both sides using

$$u = I - \mathcal{E}/R$$

$$du = dI$$

$$\int_{I(0)-\mathcal{E}/R}^{I(t)-\mathcal{E}/R} \frac{du}{u} = -\frac{R}{L}(t-0)$$

$$\ln\left(\frac{I(t)-\mathcal{E}/R}{I(0)-\mathcal{E}/R}\right) = -\frac{Rt}{L} = \ln\left(\frac{\mathcal{E}/R - I(t)}{\mathcal{E}/R - I(0)}\right)$$

taking exponential

$$(\mathcal{E}/R - I(t)) = (\mathcal{E}/R - I(0))\,e^{-Rt/L}$$

$$I(t) = \frac{\mathcal{E}}{R} - \left(\frac{\mathcal{E}}{R} - I(0)\right)e^{-Rt/L}$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right) + I(0)\,e^{-Rt/L}}$$

If the initial current is 0 then

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

in this case as t increases $I(t) \to \frac{\varepsilon}{R}$ which is what you expect from ohm's law, but it does not reach this state immediately. - if R is small or L is large it take more time to reach a steady state

If the battery is removed then

$$I(t) = I(0)\, e^{-Rt/L}$$

If $I(0)$ is ε/R (ohms law current) then

$$I(t) = \frac{\varepsilon}{R} e^{-tR/L}$$

The power given off is

$$P = I^2(t) R$$

$$= \frac{\varepsilon^2}{R} e^{-2tR/L}$$

Integrating this from $0 - \infty$ gives the energy initially stored in the inductor.

$$\text{energy} = \int_0^\infty \frac{\mathcal{E}^2}{R} e^{-\frac{2Rt}{L}} \, dt$$

Let $u = \frac{2Rt}{L}$ $\quad du = \frac{2R}{L} dt \quad dt = \frac{L}{2R} du$

$$= \int_0^\infty \frac{\mathcal{E}^2}{R}\left(\frac{L}{2R}\right) e^{-u} \, du$$

$$= \frac{1}{2}\left(\frac{\mathcal{E}}{R}\right)^2 L \left(-e^{-\infty} - (-e^{-0})\right)$$

$$\boxed{\text{energy} = \frac{1}{2} I^2(0) L}$$

since $I = \frac{\Phi}{L}$ this becomes

$$\text{energy} = \frac{1}{2} \Phi^2 \frac{1}{L}$$

for our solenoid

$$= \frac{1}{2} \frac{(B n \ell A)^2}{A \ell \mu_0 n^2} = \frac{1}{2\mu_0} B^2 A \ell$$

$$= \frac{1}{2\mu_0} B^2 \times \text{volume}$$

this suggests that like the electric field - energy is stored in the magnetic field with energy / volume given by

$$\boxed{\frac{\text{magnetic energy}}{\text{volume}} = \frac{B^2}{2\mu_0}}$$

mutual inductance

consider 2 current loops

1 2

the current in loop 1 creates a magnetic flux in loop 2. the flux in loop 2 is proportional to the current in loop 1. this gives

$$\Phi_2 = M_{21} I_1$$

similarly

$$\Phi_1 = M_{12} I_2$$

If we differentiate both sides with respect to time

$$\boxed{\begin{aligned} \mathcal{E}_2 &= -\frac{d\Phi_1}{dt} = -M_{21}\frac{dI_1}{dt} \\ \mathcal{E}_1 &= -\frac{d\Phi_1}{dt} = -M_{12}\frac{dI_2}{dt} \end{aligned}}$$

the quantity M_{12} and M_{21} are actually the same

example:

To treat this circuits we use conservation of charge and energy again

$$I_1 = I_2 + I_3$$

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0$$

$$\varepsilon - I_1 R_1 - L\frac{dI_3}{dt}$$

first eliminate I_2

$$I_2 = I_1 - I_3$$

$$\varepsilon - I_1 R_1 - (I_1 - I_3) R_2$$

eliminate I_1

$$I_1 (R_1 + R_2) = \varepsilon + I_3 R_2$$

$$I_1 = \frac{\varepsilon + I_3 R_2}{R_1 + R_2}$$

last equation

$$\boxed{\varepsilon - R_1\left(\frac{\varepsilon + I_3 R_2}{R_1 + R_2}\right) - L\frac{dI_3}{dt} = 0}$$

This gives a simple differential equation for I_3

$$\frac{dI_3}{dt} = \underbrace{\frac{1}{L}\left(\varepsilon - \frac{R_1}{R_1+R_2}\varepsilon\right)}_{A} - \underbrace{\frac{R_1R_2}{LR_1+R_2}}_{B} I_3$$

$$\frac{dI_3}{dt} = A - BI_3$$

$$\frac{dI_3}{BI_3 - A} = -dt \qquad \text{mult by } B$$

$$\frac{dI_3}{I_3 - A/B} = -B\,dt$$

$$u = I_3 - A/B \qquad du = dI_3$$

$$\int_{I_3(0)-A/B}^{I_3(t)-A/B} \frac{du}{u} = -B\int_0^t dt$$

$$\ln\left(\frac{A/B - I_3(t)}{A/B - I_3(0)}\right) = -Bt$$

$$\left(A/B - I_3(t)\right) = \left(A/B - I_3(0)\right)e^{-Bt}$$

$$\boxed{I_3(t) = A/B\left(1 - e^{-Bt}\right) + I_3(0)e^{-Bt} \qquad A = \frac{\varepsilon}{L}\frac{R_2}{R_1+R_2} \quad B = \frac{R_1R_2}{L(R_1+R_2)}}$$

$$I_1(t) = \frac{\varepsilon + I_3(t)R_2}{(R_1+R_2)}$$

$$I_2(t) = \frac{\varepsilon}{R_1+R_2} + I_3(t)\left(\frac{R_2}{R_1+R_2} - 1\right)$$

$$= \frac{\varepsilon}{R_1+R_2} - \frac{R_1}{R_1+R_2}I_3$$

Inductors in parallel and series

$$\mathcal{E} = -L \frac{dI}{dt}$$

2 inductors in series

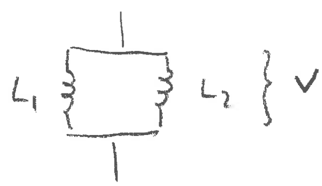
$$\left.\begin{aligned} L_1 \frac{dI}{dt} &= -V_1 \\ L_2 \frac{dI}{dt} &= -V_2 \end{aligned}\right\} \quad V = V_1 + V_2 = -\left(L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt}\right)$$

$$= -(L_1 + L_2)\frac{dI}{dt}$$

In this case the effective inductance is

$$\boxed{L = L_1 + L_2 \quad \text{inductors in series}}$$

2 inductors in parallel



$$V = L\frac{dI}{dt} = L_1 \frac{dI_1}{dt} = L_2 \frac{dI_1}{dt}$$

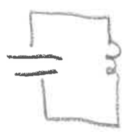
$$\frac{V}{L} = \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} = \frac{V}{L_1} + \frac{V}{L_2}$$

since V is the same

$$\boxed{\begin{aligned} &\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} \\ \text{or} \quad & L = \frac{L_1 L_2}{L_1 + L_2} \end{aligned} \quad \text{inductors in parallel}}$$

So far we have discussed RC and RL circuits. We can also consider LC and RLC circuits as well

Case of an LC circuit



The energy conservation equation gives

$$0 = -\frac{Q}{C} - L\frac{dI}{dt}$$

we note $I = \frac{dQ}{dt}$ $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ so

we can write this equation as

$$\boxed{L\frac{d^2Q}{dt^2} = -\frac{Q}{C}}$$

This should look familiar

$$m\frac{d^2x}{dt^2} = -kx$$

which is the equation for a particle of mass m attached to a spring

These equations have the same form if we replace

$m \to L$

$k \to \frac{1}{C}$

$x \to Q$

If we write this as

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC} Q$$

we know from mechanics that the general solution has the form

$$\boxed{Q(t) = Q(0) \cos\left(\frac{t}{\sqrt{LC}}\right) + I(0)\sqrt{LC} \sin\left(\frac{t}{\sqrt{LC}}\right)}$$

where $Q(0)$ is the initial charge and $I(0)$ is the initial current.

$$\boxed{I(t) = \frac{dQ}{dt} = -Q(0)\frac{1}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) + I(0) \cos\left(\frac{t}{\sqrt{LC}}\right)}$$

If at $t=0$ the capacitor is initially charged and the initial current is 0 then

$$\boxed{\begin{aligned} Q(t) &= Q(0) \cos\left(\frac{t}{\sqrt{LC}}\right) \\ I(t) &= -\frac{Q(0)}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}}$$

In the absence of any dissipation these continue to oscillate with angular frequency

$$\boxed{\omega = \frac{1}{\sqrt{LC}} = 2\pi f}$$

and period

$$\boxed{T = \frac{1}{f} = 2\pi\sqrt{LC}}$$

Alternative treatment

complex numbers

$$i = \sqrt{-1}$$

$$z = x + iy$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_2 x_1)$$

consider

$$e^{ia}$$

$$\frac{d}{dt} e^{iat} = ia e^{iat}$$

$$\frac{d^2}{dt^2} e^{iat} = (ia)^2 e^{iat} = -a^2 e^{iat}$$

we note

cos at and sin at

satisfy the same equation

these are independent solutions so we must have

$$\cos ax = \alpha e^{iax} + \beta e^{-iax}$$
$$\sin ax = \gamma e^{iax} + \delta e^{-iax}$$

for $x = 0$

$$1 = \alpha + \beta$$
$$0 = \gamma + \delta$$

differentiating

$$-a \sin ax = ia\alpha e^{iax} - ia\beta e^{-iax}$$
$$a \cos ax = i\gamma a e^{iax} - i\delta a e^{-iax}$$

setting $x = 0$

$$0 = \alpha - \beta$$
$$1 = i(\gamma - \delta)$$

this gives $\alpha = \beta = \frac{1}{2}$ $\quad \delta = -\gamma = -\frac{1}{2}$

$$\cos ax = \frac{1}{2}(e^{iax} + e^{-iax})$$
$$\sin ax = \frac{1}{2i}(e^{iax} - e^{-iax})$$

we can also invert these to g

$$e^{iax} = \cos ax + i \sin ax$$
$$e^{-iax} = \cos ax - i \sin ax$$