Lecture 27

Last time

Inductors

$$\Phi = LI$$

$$\frac{d\Phi}{dt} = -\mathcal{E} = L\frac{dI}{dt}$$

Long solenoid

$$L = \mu_0 n^2 A L$$

From

$$B = \mu_0 n I$$

$$\Phi = B n L A = \mu_0 n^2 L A I$$

RL circuits

$$\mathcal{E} - L\frac{dI}{dt} - RI = 0 \qquad \text{or} \qquad L\frac{dI}{dt} = -RI$$

$$\frac{dI}{dt} = \frac{1}{L}(\mathcal{E} - IR) \qquad\qquad \frac{dI}{dt} = -\frac{R}{L}I$$

$$= -\frac{R}{L}\left(I - \frac{\mathcal{E}}{R}\right) \qquad\qquad I(t) = I(0)\,e^{-\frac{R}{L}t}$$

$$I(t) = \frac{\mathcal{E}}{R} + \left(I(0) - \frac{\mathcal{E}}{R}\right)e^{-\frac{R}{L}t}$$

We used this to show

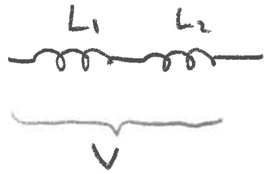
$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

and the energy in the inductor

$$\mathcal{E} = \frac{1}{2} L I^2$$

complex circuits with inductors

inductors in series

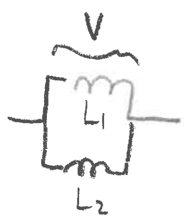


$$V = V_1 + V_2 = -L_1 \frac{dI}{dt} - L_2 \frac{dI}{dt} = -L \frac{dI}{dt}$$

canceling $-\frac{dI}{dt}$ gives

$$\boxed{L = L_1 + L_2 \quad \text{inductors in series}}$$

inductors in parallel



$$I = I_1 + I_2$$

$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$-\frac{V}{L} = -\frac{V}{L_1} - \frac{V}{L_2}$$

dividing by $-V$

$$\boxed{\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{inductors in parallel}}$$

A related concept is mutual inductance

The current in loop 1 creates a field that results in a flux in loop 2

$$\Phi_2 = M_{21} I_1$$

similarly

$$\Phi_1 = M_{12} I_2$$

M_{ij} is called the mutual inductance;

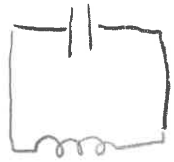
it turns out - but is not obvious that $M_{12} = M_{21} = M$.

$$\mathcal{E}_2 = - M_{21} \frac{dI_1}{dt}$$

$$\mathcal{E}_1 = - M_{12} \frac{dI_2}{dt}$$

In addition to RC and RL circuits we can consider LC and RLC circuits

LC



In this case assume that the capacitor initially has charge Q, and there is no current

$$Q(0) = Q$$
$$I(0) = 0$$

conservation of energy gives

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0$$

since $I = \frac{dQ}{dt}$ $\quad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$

putting everything together

$$\boxed{L\frac{d^2Q}{dt^2} = -\frac{Q}{C} \qquad Q(0) = Q \qquad \frac{dQ}{dt}(0) = 0}$$

To understand how to solve this consider

$$\boxed{m\frac{d^2x}{dt^2} = -kx \qquad x(0) = x_0 \qquad V(0) = \frac{dx}{dt}(0) = 0}$$

this is the equation for a mass m attached to a spring with spring constant k

$m \rightarrow L$

$x \rightarrow Q$ $\qquad \frac{dx}{dt} \rightarrow \frac{dQ}{dt} = I$

$k \rightarrow \frac{1}{C}$

solution

$$x(t) = x(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right) + v(0)\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$\frac{dx}{dt}(t) = v(t) = -x(0)\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) + v(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

making the substitutions

$$\boxed{\begin{aligned} Q(t) &= Q(0)\cos\left(\sqrt{\tfrac{1}{LC}}\,t\right) + I(0)\sqrt{LC}\sin\left(\tfrac{1}{\sqrt{LC}}\,t\right) \\ I(t) &= -\frac{Q(0)}{\sqrt{LC}}\sin\left(\sqrt{\tfrac{1}{LC}}\,t\right) + I(0)\cos\left(\sqrt{\tfrac{1}{LC}}\,t\right) \end{aligned}}$$

note the energy in the capacitor is $\frac{Q^2}{2C}$ while the energy in the inductor is $\frac{1}{2}LI^2$

$$\frac{Q^2}{2C} + \frac{1}{2}LI^2 =$$

$$\frac{\left(Q(0)\cos + I(0)\sqrt{LC}\sin\right)^2}{2C} + \frac{L}{2}\left(-\frac{Q(0)}{\sqrt{LC}}\sin + I(0)\cos\right)^2$$

$$Q(0)^2\left(\frac{1}{2C}\cos^2 + \frac{1}{2C}\sin^2\right) +$$

$$\frac{1}{2}I(0)^2L\left(\sin^2 + \cos^2\right) +$$

$$\frac{2Q(0)I(0)}{2}\sqrt{\frac{L}{C}}\sin\cos - \frac{2Q(0)I(0)}{2}\sqrt{\frac{L}{C}}\sin\cos =$$

$$\frac{Q(0)^2}{2C} + \frac{1}{2}LI(0)^2$$

This means that the total energy at any given time is conserved.

When $I(t) = 0$ all of the energy is stored in the electric field between the plates of the capacitor

When $Q(t) = 0$ all of the energy is stored in the magnetic field inside of the inductor.

in general the energy is shared between the capacitor and inductor

there is another way to represent solutions of equations of this type.

note $\frac{d^2}{dx^2} e^{ax} = a^2 e^{ax}$

$\frac{d^2}{dx^2} e^{-ax} = (-a)^2 e^{-ax}$

we see e^{ax} and e^{-ax} are solutions of

$$\boxed{\frac{d^2 f}{dx^2} = a^2 f}$$

To extend this to

$$\boxed{\frac{d^2 f}{dx^2} = -a^2 f}$$

we look for a solution to

$$\boxed{c^2 = -1}$$

since real numbers have non negative squares, there are no real solutions to this equation.

If we could find such a solution c then

$$\boxed{\begin{aligned} \frac{d}{dx} e^{acx} &= ac\, e^{acx} \\ \frac{d^2}{dx^2} e^{acx} &= a^2c^2 e^{acx} = -a^2 e^{acx} \end{aligned}}$$

The number $c \equiv i$ is an imaginary number. In this case the independent solutions are

$$e^{iax} \qquad e^{-iax}$$

how are they related to $\cos(ax)$ and $\sin(ax)$

$$e^{iax} = \alpha \cos ax + \beta \sin ax$$

$$e^{-iax} = \gamma \cos ax + \delta \sin ax$$

(1) set $x = 0$

$$1 = \alpha$$

$$1 = \gamma$$

(2) differentiate, set $x = 0$

$$i \cdot a = -\alpha a \sin ax + \beta a \cos ax$$

$$i = \beta$$

$$-i\, a = -\gamma a \sin ax + \delta a \cos ax$$

$$-i = \delta$$

$$\boxed{\begin{aligned} e^{iax} &= \cos ax + i \sin ax \\ e^{-iax} &= \cos ax - i \sin ax \end{aligned}}$$

we can invert this

$$2 \cos ax = e^{iax} + e^{-iax}$$

$$\boxed{\cos ax = \frac{1}{2}\left(e^{iax} + e^{-iax}\right)}$$

$$2 i \sin ax = \frac{1}{2}\left(e^{iax} - e^{-iax}\right)$$

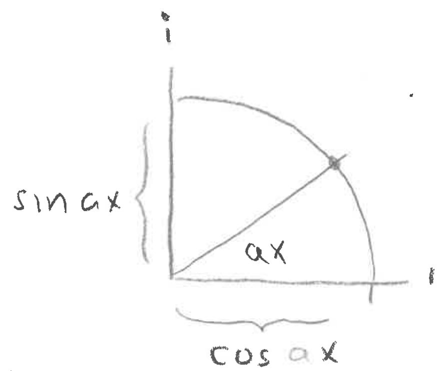
$$\boxed{\sin ax = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)}$$

some general remarks

$$z = x + i y$$

$$z^* = x - i y$$

$$\frac{1}{i} = -i$$



$$e^{i\frac{\pi}{2}} = i \qquad e^{3\frac{\pi}{2} i} = -i$$

$$e^{i\pi} = -1 \qquad e^{2\pi i} = 1$$

quadratic formula

$$a x^2 + b x + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

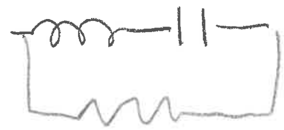
if $4ac > b^2$ then

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

in this case the roots are complex

Why bother.

consider an RLC circuit



$$\boxed{-L\frac{dI}{dt} - IR - \frac{Q}{C} = 0}$$

If we note $I = \frac{dQ}{dt}$ $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$

this becomes

$$\boxed{-L\frac{d^2Q}{dt^2} - R\frac{dQ}{dt} - \frac{1}{C}Q = 0}$$

To solve this we try a solution of the form

$$Q(t) = (\text{const})\, e^{\alpha t}$$

$$Q = Q_0 e^{\alpha t}$$

$$\frac{dQ}{dt} = Q_0 \alpha e^{\alpha t}$$

$$\frac{d^2Q}{dt^2} = Q_0 \alpha^2 e^{\alpha t}$$

using these quantities in the differential equation gives

$$-L\alpha^2 Q_0 e^{\alpha t} - R\alpha Q_0 e^{\alpha t} - \frac{1}{C} Q_0 e^{\alpha t} = 0$$

dividing by $Q_0 e^{\alpha t}$ gives an equation for α

$$-L\alpha^2 - R\alpha - \frac{1}{C} = 0$$

this is a quadratic equation. The roots are

$$\alpha = \frac{R \pm \sqrt{R^2 - \frac{4L}{C}}}{-2L}$$

$$= -\frac{R}{2L}\left(1 \pm \sqrt{1 - \frac{4L}{R^2C}}\right)$$

there are 3 possibilities

(1) $1 > \frac{4L}{R^2C}$

(2) $\frac{4L}{R^2C} > 1$

(3) $\frac{4L}{R^2C} = 1$

In the first 2 case there are 2 solutions

$$e^{-\frac{R}{2L}\left(1+\sqrt{1-\frac{4L}{R^2C}}\right)t}$$

$$e^{-\frac{R}{2L}\left(1-\sqrt{1-\frac{4L}{R^2C}}\right)t}$$

in this case both numbers in the exponent are positive becaus

$$1 > \sqrt{1-\frac{4L}{R^2C}}$$

The general solution has the form

$$Q(t) = A_+ e^{-\frac{R}{2L}\left(1+\sqrt{1-\frac{4L}{R^2C}}\right)t} + A_- e^{-\frac{R}{2L}\left(1+\sqrt{1-\frac{4L}{R^2C}}\right)t}$$

in this case after a long time both solutions damp to 0

A_+ and A_-

are determined from the initial charge and current

In the second case the independent solutions look like

$$G(t) = e^{-\frac{R}{2L}t}\left(B_+ e^{i\frac{R}{2L}\sqrt{\frac{4L}{R^2C}-1}\,t} + B_- e^{-i\frac{R}{2}\left(\sqrt{\frac{4L}{R^2C}-1}\right)t}\right)$$

$$e^{-\frac{R}{2L}t}\left(B_+ e^{i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t} + B_- e^{-i\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t}\right)$$

This solution has 2 components - there is a part that decays like $e^{-\frac{R}{2L}t}$ and a part that oscillates with angular frequency $\omega = \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$

This frequency is reduced relative to the frequency of an LC circuit as $\frac{R}{2L}$ increases the frequency $\to 0$

This could also be written as

$$e^{-\frac{R}{2L}t}\left(C^+\cos\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t\right) + C^-\sin\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\,t\right)\right)$$

where $C^+ = \frac{1}{2}(B_+ + B_-)$

$C^- = \frac{1}{2i}(B_+ - B_-)$