Lecture 27

LC circuits

$$V = -L\frac{dI}{dt} = -L\frac{d^2Q}{dt^2}$$

$$V = -\frac{Q}{C}$$

loop equation

$$-L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0 \qquad \omega = \sqrt{\frac{1}{LC}}$$

looks like

$$-m\frac{d^2x}{dt^2} - kx = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

in this case by analogy the solution has the form

$$\boxed{\begin{aligned} Q(t) &= Q(0)\cos\left(\sqrt{\tfrac{1}{LC}}\,t\right) + I(0)\sqrt{LC}\,\sin\left(\sqrt{\tfrac{1}{LC}}\,t\right) \\ I(t) &= -\frac{Q(0)}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}\,t\right) + I(0)\cos\left(\sqrt{\tfrac{1}{LC}}\,t\right) \end{aligned}}$$

these solutions follow because

$$\frac{d^2}{dx^2}\cos(ax) = -a^2\cos(ax)$$

$$\frac{d^2}{dt^2}\sin(ax) = -a^2\sin(ax)$$

These are 2 independent solutions, it follows that

$$\alpha \sin(ax) + \beta \cos(ax)$$

is also a solution for any α, β. In our case α, β are fixed by specifying the initial charge and current. Once these are fixed $\frac{dI}{dt}(0)$ is fixed by the differential equation

There is another more useful way to represent these solutions

consider
$$\frac{d}{dx} e^{\pm at} = \pm a e^{\pm at}$$
$$\frac{d^2}{dx^2} e^{\pm at} = (\pm a)^2 e^{\pm at} = a^2 e^{\pm at}$$

for real numbers $a^2 \geq 0$. If we introduce a new number c with $c^2 = -1$ then

$$\frac{d^2}{dx^2} e^{\pm ac} = (a^2c^2) e^{\pm ac} = -a^2 e^{\pm ac}$$

it is conventional to call $c \equiv i$
defined as $\sqrt{-1} = i$

we define a general complex
number by

$$z = x + iy$$

where x and y are ordinary
real numbers

note

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

we define the complex conjugate
of z by

$$z^* = (x - iy)$$

note

$$zz^* = (x+iy)(x-iy) = x^2 + (iy)(-iy) = x^2 + y^2$$

and

$$|z| = \sqrt{zz^*} = \sqrt{x^2+y^2}$$

(called the modulus of the
complex number z)

for $z = x + iy$

x is called the real part of z

y is called the imaginary part of z

(note y is still a real number)

It follow that

$$e^{i\frac{1}{\sqrt{LC}}t} \qquad e^{-i\frac{1}{\sqrt{LC}}t}$$

are also solutions of

$$L\frac{d^2Q}{dt^2} = -\frac{1}{C}Q$$

This means

$$e^{iat} = \alpha \cos at + \beta \sin at$$

$$e^{-iat} = \gamma \cos at + \delta \sin at.$$

To find α β γ δ set $t = 0$

$$\alpha = \gamma = 1$$

next differentiate and set $t = 0$

$$\left.\begin{aligned} iae^{iat} &= -\alpha a \sin at + \beta a \cos at \\ -iae^{iat} &= -\gamma a \sin at \quad \delta a \cos at \end{aligned}\right\} \Rightarrow$$

$i = \beta, \quad -i = \delta$

which gives

$$e^{iat} = \cos(at) + i\sin(at)$$

$$e^{-iat} = \cos(at) - i\sin(at)$$

inverting

$$\cos(at) = \frac{1}{2}\left(e^{iat} + e^{-iat}\right)$$

$$\sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right)$$

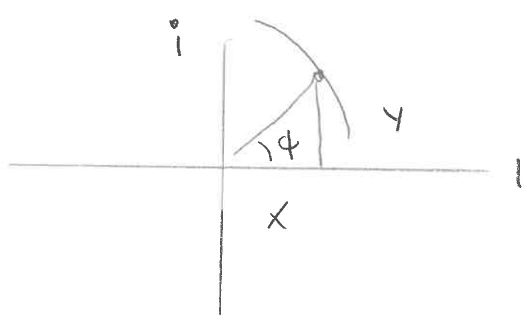
how do you divide complex number

$$\frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}\,\frac{x_2 - iy_2}{x_2 - iy_2} = \frac{Z_1 Z_2^*}{Z_2 Z_2^*} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{\sqrt{x_2^2 + y_2^2}}$$

consider

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$= x + iy$$



if $\phi = \frac{\pi}{2}$ then

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right) = i$$

$$e^{i\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

finally note that if we want to find the roots of

$$ax^2 + bx + c$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

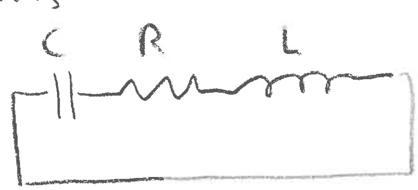
if $b^2 \geq 4ac$ this is real

if $b^2 < 4ac$ we can write this as

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

in which case there are 2 complex roots

We will use this to treat RLC circuits



The loop equation gives

$$-\frac{Q}{C} - IR - L\frac{dI}{dt} = 0$$

expressing everything in terms of Q gives

$$L\frac{d^2Q}{dt^2} + I\frac{dQ}{dt} + \frac{Q}{C} = 0$$

If we assume a solution of the form

$$Q(t) = e^{zt}$$

and insert this into the differential equation we get

$$\left(z^2L + Rz + \frac{1}{C}\right)e^{zt} = 0$$

this gives a quadratic equation for z

$$\left(z^2L + Rz + \frac{1}{C}\right) = 0$$

$$z = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

$$\boxed{z = -\frac{R}{2L} \pm \frac{R}{2L}\sqrt{1 - \frac{4L}{R^2C}}}$$

There are 3 cases

① $1 > \frac{4L}{R^2C}$

then both

$$z_+ = -\frac{R}{2L} - \frac{R}{2L}\sqrt{1-\frac{4L}{R^2C}} < 0, \text{ real}$$

$$z_- = -\frac{R}{2L} + \frac{R}{2L}\sqrt{1-\frac{4L}{R^2C}} < 0, \text{ real}$$

the general solution has the form

$$\boxed{Q(t) = \alpha e^{z_+ t} + \beta e^{z_- t}}$$

which describes two exponentially damped solutions

$$Q(0) = \alpha + \beta$$

$$I(0) = \frac{dQ}{dt}(0) = \alpha z_+ + \beta z_-$$

$$\beta = Q(0) - \alpha$$

$$I(0) = \alpha z_+ + (Q(0) - \alpha) z_-$$

$$\alpha(z_+ - z_-) = I(0) - Q(0) z_-$$

$$\alpha = \frac{I(0) - Q(0) z_-}{z_+ - z_-}$$

$$\beta = Q(0) - \alpha = \frac{Q(0)(z_+ - z_-) - I(0) + Q(0) z_-}{z_+ - z_-}$$

$$= \frac{Q(0) z_+ - I(0)}{z_+ - z_-}$$

the second case is

$$\frac{4L}{R^2C} > 1$$

then

$$z_\pm = -\frac{R}{2L} \pm i\frac{R}{2L}\sqrt{\frac{4L}{R^2C}-1}$$

$$= -a \pm ib$$

$$a = \frac{R}{2L} \quad b = \frac{R}{2L}\sqrt{\frac{4L}{R^2C}-1}$$

$$Q(t) = e^{-at}\left(\alpha e^{ibt} + \beta e^{-ibt}\right)$$

this could also be expressed in terms of $\sin bt$ and $\cos(bt)$

$$Q(t) = e^{-at}\left(\alpha' \cos bt + \beta' \sin bt\right)$$

$$I(t) = -a e^{-at}\left(\alpha' \cos bt + \beta' \sin bt\right)$$
$$+ b e^{-at}\left(-\alpha' \sin bt + \beta' \cos bt\right)$$

setting $t = 0$

$$Q(0) = \alpha'$$

$$I(0) = -a\alpha' + b\beta' = -aQ(0) + b\beta'$$

$$\beta' = \frac{1}{b}\left(I(0) + aQ(0)\right)$$

In this case

$$Q(t) = e^{-at}\left(Q(0)\cos(bt) + \frac{1}{b}\left(I(0) + a\,Q(0)\right)\sin(bt)\right)$$

where the current can be determined by differentiating

In this case there are oscillations with

$$\omega = \frac{R}{2L}\sqrt{\frac{4L}{R^2C} - 1} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} < \sqrt{\frac{1}{LC}}$$

reduced relative to a circuit with $R=0$. – and an amplitude that decreases like $e^{-\frac{R}{2L}t}$

There is one more case – when

$$\frac{4L}{R^2C} = 1$$

In this case the square root term vanishes. It looks like there is only 1 independent solution, but we will see that there is a second independent solution

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

consider a solution of the form

$Q = te^{at}$

$$\frac{dQ}{dt} = e^{at} + tae^{at}$$

$$\frac{d^2Q}{dt^2} = ae^{at} + ae^{at} + ta^2e^{at}$$

using this in the differential equation

$$\left[L(a+a+ta^2) + R(1+at) + \frac{1}{C}t\right]e^{at} = 0$$

$$\left[\underbrace{(L(2a)+R)}_{} + t\underbrace{(La^2+Ra+\tfrac{1}{C})}_{\text{original quadratic equation for } a}\right]e^{at} = 0$$

$$L\left(2\left(-\frac{R}{2L}\right)\right)+R$$

$$R-R=0$$

$$a = -\frac{R}{2L}$$

in this case the solutions are

$$Q(t) = \alpha e^{-\frac{R}{2L}t} + \beta t e^{-\frac{R}{2L}t}$$

at time 0

$$Q(0) = \alpha$$

$$I(0) = -\alpha\left(\frac{R}{2L}\right) + \beta = 0$$

$$\beta = I(0) + Q(0)R/2L$$

In this case

$$\boxed{Q(t) = Q(0)\, e^{-\frac{R}{2L}t} + \left(I(0) + Q(0)\frac{R}{2L}\right) t\, e^{-\frac{R}{2L}t}}$$

In this case both solutions are damping

where does the other solution come from

$$\begin{pmatrix} \frac{dQ}{dt} \\ \frac{dI}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{R}{L} & -\frac{1}{LC} \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} = \underline{\underline{M}} \begin{pmatrix} Q \\ I \end{pmatrix}$$

$$\frac{d^n}{dt^n} \begin{pmatrix} Q \\ I \end{pmatrix} = \underline{\underline{M}}^n \begin{pmatrix} Q \\ I \end{pmatrix}$$

Taylor

$$\begin{pmatrix} Q(t) \\ I(t) \end{pmatrix} = \sum \frac{1}{n!} \underline{\underline{M}}^n t^n \begin{pmatrix} Q(0) \\ I(0) \end{pmatrix} = e^{\underline{\underline{M}}t} \begin{pmatrix} Q(0) \\ I(0) \end{pmatrix}$$

for matrices if $\underline{\underline{M}}^2 = 0$

$$e^{\underline{\underline{M}}t} = (1 + \underline{\underline{M}}t)$$