Lecture 27

LC circuit

\[ V = -L \frac{dx}{dt} = -L \frac{d^2q}{dt^2} \]

\[ V = -\frac{q}{C} \]

Loop equation

\[ -L \frac{d^2q}{dt^2} - \frac{q}{C} = 0 \quad \omega = \sqrt{\frac{1}{LC}} \]

Looks like

\[ -m \frac{d^3x}{dt^3} - kx = 0 \quad \omega = \sqrt{\frac{k}{m}} \]

In this case by analogy the solution has the form

\[ q(t) = Q(c) \cos \left( \sqrt{\frac{1}{LC}} t \right) + I(0) \sqrt{\frac{1}{LC}} \sin \left( \sqrt{\frac{1}{LC}} t \right) \]

\[ i(t) = -\frac{Q(c)}{\sqrt{LC}} \sin \left( \sqrt{\frac{1}{LC}} t \right) + I(0) \cos \left( \sqrt{\frac{1}{LC}} t \right) \]

These solutions follow because

\[ \frac{d^2}{dx^2} \cos(ax) = -a^2 \cos(ax) \]

\[ \frac{d^2}{dt^2} \sin(ax) = -a^2 \sin(ax) \]
These are 2 independent solutions. It follows that
\[ \alpha \sin(\alpha x) + \beta \cos(\alpha x) \]
is also a solution for any \( \alpha, \beta \).
In our case \( \alpha, \beta \) are fixed by specifying the initial charge and current. One these are fixed
\[ \frac{dI}{dt}(c) \]
is fixed by the differential equation.

Here is another more useful way to represent these solutions.

Consider
\[ \frac{d}{dx} e^{\pm \alpha t} = \mp \alpha e^{\pm \alpha t} \]
\[ \frac{d^2}{dx^2} e^{\pm \alpha t} = (\pm \alpha)^2 e^{\pm \alpha t} = \alpha^2 e^{\pm \alpha t} \]
for real numbers \( \alpha^2 \geq 0 \). If we introduce a new number \( c \) with \( c^2 = -1 \) then
\[ \frac{d^2}{dx^2} e^{\pm \alpha c} = (\alpha c^2) e^{\pm \alpha c} = -\alpha^2 e^{\pm \alpha c} \]
It is conventional to call $c = i$ defined as $\sqrt{-1} = i$

We define a general complex number by

$$Z = x + iy$$

where $x$ and $y$ are ordinary real numbers.

Note

$$Z_1 \cdot Z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

We define the complex conjugate of $Z$ by

$$Z^* = (x - iy)$$

Note

$$ZZ^* = (x + iy)(x - iy) = x^2 + (iy)(-iy) = x^2 + y^2$$

and

$$|Z| = \sqrt{ZZ^*} = \sqrt{x^2 + y^2}$$

(called the modulus of the complex number $Z$)
for \( z = x + iy \)

\( x \) is called the real part of \( z \)

\( y \) is called the imaginary part of \( z \)

(Note \( y \) is still a real number)

It follows that

\[
\begin{align*}
i \frac{d}{dt} t & \quad -i \frac{d}{dt} t \\
e^t & \quad e^t
\end{align*}
\]

are also solutions of

\[
L \frac{d^2 q}{dt^2} = -\frac{1}{c} q
\]

This means

\[
\begin{align*}
&i \alpha \ e^t = \alpha \cos \omega t + \beta \sin \omega t \\
&-i \alpha \ e^{-t} = \gamma \cos \omega t + \delta \sin \omega t.
\end{align*}
\]

To find \( \alpha, \beta, \gamma, \delta \) set \( t = 0 \)

\[
\alpha = \gamma = 1
\]

Next differentiate twice and set \( t = 0 \)

\[
\begin{align*}
&i \alpha e^t = -\alpha \alpha \sin \omega t + \beta \alpha \cos \omega t \\
&-i \alpha e^{-t} = -\gamma \alpha \sin \omega t + \delta \alpha \cos \omega t
\end{align*}
\]
\[ i = \beta \iff -i = -\beta \]

which gives
\[ e^{i \alpha t} = \cos(\alpha t) + i \sin(\alpha t) \]
\[ e^{-i \alpha t} = \cos(\alpha t) - i \sin(\alpha t) \]

Inventing
\[ \cos(\alpha t) = \frac{1}{2} (e^{i \alpha t} + e^{-i \alpha t}) \]
\[ \sin(\alpha t) = \frac{1}{2i} (e^{i \alpha t} - e^{-i \alpha t}) \]

how do you divide complex number
\[
\frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{Z_1 \overline{Z_2}}{Z_2 \overline{Z_2}} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{\sqrt{x_2^2 + y_2^2}}
\]

Consider
\[ e^{i \phi} = \cos \phi + i \sin \phi \]
\[ = x + iy \]

![Diagram](image)
If \( \phi = \frac{\pi}{2} \) then
\[
\begin{align*}
  e^{i \frac{\pi}{2}} &= \cos \frac{\pi}{2} + i \sin \left( \frac{\pi}{2} \right) = i \\
  e^{i \frac{3\pi}{2}} &= \cos \left( \frac{3\pi}{2} \right) + i \sin \left( \frac{3\pi}{2} \right) = -i
\end{align*}
\]

Finally note that if we want to find the roots of
\[
ax^2 + bx + c = 0
\]
\[
X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

If \( b^2 \geq 4ac \) this is real
If \( b^2 < 4ac \) we can write this as
\[
X = \frac{-b \pm i \sqrt{4ac - b^2}}{2a}
\]
in which case there are 2 complex roots.

We will use this to treat RLC circuits:

\[
\begin{array}{c}
C \quad R \quad L
\end{array}
\]
The loop equation gives

$$-\frac{Q}{C} - IR - L \frac{dI}{dt} = 0$$

Expressing everything in terms of $Q$ gives

$$L \frac{d^2Q}{dt^2} + I \frac{dQ}{dt} + \frac{Q}{C} = 0$$

If we assume a solution of the form

$$Q(t) = et^k$$

and insert this into the differential equation we get

$$(Z^2L + RZ + \frac{1}{C})et^k = 0$$

This gives a quadratic equation for $Z$

$$(Z^2L + RZ + \frac{1}{C}) = 0$$

$$Z = \frac{-R \pm \sqrt{R^2 - 4L}}{2L}$$

$$Z = \frac{-R \pm \frac{R}{2L} \sqrt{1 - \frac{4L}{R^2}}}{2L}$$
new are 3 cases

1. \( l > \frac{4L}{Rc} \)
   
   then both
   
   \[ z_+ = -\frac{R}{2L} - \frac{R}{2L} \sqrt{1 - \frac{4L}{Rc}} < 0, \text{ real} \]
   
   \[ z_- = \frac{R}{2L} + \frac{R}{2L} \sqrt{1 - \frac{4L}{Rc}} < 0, \text{ real} \]

   the general solution has the form
   
   \[ q(t) = \alpha e^{z_+ t} + \beta e^{z_- t} \]

   which describes two exponentially damped solutions

   \[ \theta(l) = \alpha + \beta \]

   \[ I(l) = \frac{d\theta}{dt}(l) = \alpha z_+ + \beta z_- \]

   \[ \beta = \theta(l) - \alpha \]

   \[ I(l) = \alpha z_+ + (\theta(l) - \alpha) z_- \]

   \[ \alpha(z_+ - z_-) = I(l) - \theta(l) z_- \]

   \[ \alpha = \frac{I(l) - \theta(l) z_-}{z_+ - z_-} \]

   \[ \beta = \theta(l) - \alpha = \frac{\theta(l)(z_+ - z_-) - I(l) + \theta(l) z_-}{z_+ - z_-} \]

   \[ = \frac{\theta(l) z_+ - I(l)}{z_+ - z_-} \]
The second case is
\[ \frac{4L}{R^3 c} > 1 \]

Then
\[ Z_\pm = -\frac{R}{2L} \pm i \frac{R}{2L} \sqrt{\frac{4L}{R^3 c} - 1} \]
\[ = -a \pm ib \]
\[ a = \frac{R}{2L} \quad b = \frac{R}{2L} \sqrt{\frac{4L}{R^3 c} - 1} \]

\[ \Theta(t) = e^{-(\alpha + \beta t)} \left( e^{i\beta t} + e^{-i\beta t} \right) \]

This could also be expressed in terms of \( \sin \beta t \) and \( \cos \beta t \):
\[ \Theta(t) = e^{-\alpha t} \left( \alpha' \cos \beta t + \beta' \sin \beta t \right) \]
\[ \Theta(t) = -a e^{-\alpha t} \left( \alpha' \cos \beta t + \beta' \sin \beta t \right) \]
\[ + b e^{-\alpha t} \left( -\alpha' \sin \beta t + \beta' \cos \beta t \right) \]

Setting \( t = 0 \):
\[ \Theta(0) = \alpha' \]
\[ = -a \alpha' + b \beta' = -a \Theta(0) + b \beta' \]
\[ \beta' = \frac{1}{b} (\Theta(0) + a \Theta(0)) \]
In this case

$$Q(t) = e^{at} \left( Q(0) \cos (bt) + \frac{1}{b} (I(0) + G Q(0)) \sin (bt) \right)$$

where the current can be determined by differentiating.

In this case there are oscillations with

$$\omega = \frac{R}{2L} \sqrt{\frac{4L}{R^2 c}} - 1 = \sqrt{\frac{1}{4c} - \frac{R^2}{4L^2}} < \sqrt{\frac{1}{4c}}$$

reduced relative to a circuit with $R=0$, and an amplitude that decreases like $e^{-\frac{R^2}{2L} t}$

There is one more case when

$$\frac{4L}{R^2 c} = 1$$

In this case the square root term vanishes. It looks like there is only 1 independent solution, but we will see that there is a second independent solution.
\[ L \frac{d^2 \theta}{dt^2} + R \frac{d\theta}{dt} + \frac{1}{\ell} \theta = 0 \]

Consider a solution of the form

\[ \theta = te^{at} \]

\[ \frac{d\theta}{dt} = e^{at} + tae^{at} \]

\[ \frac{d^2 \theta}{dt^2} = ae^{at} + ae^{at} + tae^{at} \]

Using this in the differential equation

\[ L (a^2 + ta^2) + R (1 + at) + \frac{1}{\ell} t e^{at} = 0 \]

\[ \left[ (L (2a + R)) + t (L a^2 + Ra + \frac{1}{\ell}) \right] e^{at} = 0 \]

\[ L \left( \frac{2L - R}{2L} + R \right) + R - R = 0 \]

Original quadratic equation for \( a \)

\[ a = -\frac{R}{2L} \]

In this case the solution is

\[ \theta(t) = e^{-\frac{R}{2L}t} + \beta t e^{-\frac{R}{2L}t} \]

At time 0

\[ \theta(0) = \alpha \]

\[ I(0) = -\alpha \left( \frac{R}{2L} \right) + \beta = 0 \]

\[ \beta = I(0) + \theta(0) \frac{R}{2L} \]
In this case both solutions are damping.

where does the other solution come from

\[
\begin{pmatrix}
\frac{dQ}{dt} \\
\frac{dI}{dt}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\frac{R}{L} - \frac{1}{LC}
\end{pmatrix} \begin{pmatrix}
Q \\
I
\end{pmatrix} = M \begin{pmatrix}
Q \\
I
\end{pmatrix}
\]

\[
\frac{d^n}{dt^n} \begin{pmatrix}
Q \\
I
\end{pmatrix} = M^n \begin{pmatrix}
Q \\
I
\end{pmatrix}
\]

Taylor

\[
\begin{pmatrix}
Q(t) \\
I(t)
\end{pmatrix} = \frac{1}{0!} M^n t^n \begin{pmatrix}
Q(0) \\
I(0)
\end{pmatrix} = e^{Mt} \begin{pmatrix}
Q(0) \\
I(0)
\end{pmatrix}
\]

In matrix if \( M^2 = 0 \)

\[
e^{Mt} = (I + Mt)
\]