Lecture 28

Last time

LRC circuits

$$-L\frac{dI}{dt} - RI - \frac{Q}{C} = 0$$

use

$$I(t) = \frac{dQ}{dt}, \quad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

$$-L\frac{d^2Q}{dt^2} - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

we looked for solutions of the general form e^{at} where a could be a complex number. Using this solution in the equation gave

$$\left(-La^2 - Ra - \frac{1}{C}\right)e^{at} = 0$$

this will vanish for all time if a is a root of the quadratic equation.

$$La^2 + Ra + \frac{1}{C} = 0$$

using the quadratic formula gives

$$a = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

$$= -\frac{R}{2L}\left(1 \pm \sqrt{1 - 4\frac{L}{R^2C}}\right) = a_\pm$$

this gives 2 independent solutions

$$Q(t) = Q_+ e^{a_+ t} + Q_- e^{a_- t}$$

$$I(t) = Q_+ a_+ e^{a_+ t} + Q_- a_- e^{a_- t}$$

we can express both of these solutions in terms of the initial charge on the capacitor and current through the resistor

$$Q(0) = Q_+ + Q_-$$

$$I(0) = Q_+ a_+ + Q_- a_-$$

using

$$Q_- = Q(0) - Q_+$$

$$I(0) = Q_+ a_+ + (Q(0) - Q_+) a_-$$

$$= Q_+(a_+ - a_-) + a_- Q(0)$$

$$Q_+ = \frac{I(0) - a_- Q(0)}{a_+ - a_-}$$

$$Q_- = Q(0) - \frac{I(0) - a_- Q(0)}{a_+ - a_-}$$

$$= \frac{Q(0)\, a_+ - I(0)}{a_+ - a_-}$$

this gives

$$Q(t) = \frac{I(0) - a_- Q(0)}{a_+ - a_-} e^{a_+ t} + \frac{Q(0) a_+ - I(0)}{a_+ - a_-} e^{a_- t}$$

$$I(t) = \frac{I(0) - a_- Q(0)}{a_+ - a_-} a_+ e^{a_+ t} + \frac{Q(0) a_+ - I(0)}{a_+ - a_-} a_- e^{a_- t}$$

where

$$a_\pm = -\frac{R}{2L}\left(1 \pm \sqrt{1 - \frac{4L}{R^2 C}}\right)$$

As mentioned earlier - there are three types of time dependence, depending on

(1) $0 < 1 - \frac{4L}{R^2C} < 1$

(2) $\frac{4L}{R^2C} > 1$

(3) $\frac{4L}{R^2C} = 1$

In case 1 the quantity in the square root is positive so $a_\pm$ are <u>both</u> <u>negative</u> <u>real</u> numbers

this means both $Q(t)$ and $I(t)$ go smoothly to 0 without any oscillations case 2 $\quad \frac{4L}{R^2C} > 1$ then

$$a_\pm = -\frac{R}{2L}\left(1 \pm i\sqrt{\frac{4L}{R^2C}-1}\right)$$

in this case the solutions have the form

$$e^{-\frac{R}{2L}} e^{\mp i\sqrt{\frac{4L}{R^2C}-1}\, t}$$

$$e^{-\frac{R}{2L}}\left(\cos\left(\sqrt{\frac{4L}{R^2C}-1}\right)t \mp i\sin\left(\sqrt{\frac{4L}{R^2C}-1}\, t\right)\right)$$

This has the form of a decaying exponential multiplied by an oscillating amplitude with angular frequency $\omega = \sqrt{\frac{4L}{R^2C}-1}$

In this case we could also write the general solution as

$$e^{-\frac{R}{2L}t}\left(A\cos\left(\sqrt{\frac{4L}{R^2C}-1}\, t\right) + B\sin\left(\sqrt{\frac{4L}{R^2C}-1}\, t\right)\right)$$

$$= e^{-\frac{R}{2L}t}\left(C\cos\left(\sqrt{\frac{4L}{R^2C}-1}\, t + \phi\right)\right)$$

in all case either $a_+ a_-$, A, B or C, ϕ can be chosen to get the correct initial charge and current.

These can be related using

$$\cos(\omega t+\phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$$

The third case is more interesting

In that case $1 = \frac{4L}{R^2 C}$ $\quad a = e^{-\frac{R}{2L}t}$

The independent solutions are

$$e^{-\frac{R}{2L}t} \qquad t e^{-\frac{R}{2L}t}$$

To check this for $e^{-\frac{R}{2L}t}$

$$L\left(\frac{R^2}{4L^2}\right) + R\left(-\frac{R}{2L}\right) + \frac{1}{C} = 0 \quad \Rightarrow$$

$$\frac{R^2}{4C} = 1$$

for $t e^{-\frac{R}{2L}t}$

$$\frac{d}{dt}: \quad -\frac{R}{2L} t e^{-\frac{R}{2L}} + e^{-\frac{R}{2L}t}$$

$$\frac{d^2}{dt^2}: \quad -\frac{R}{2L} e^{-\frac{R}{2L}} + \frac{R^2}{4L^2} t e^{-\frac{R}{2L}} - \frac{R}{2L} e^{-\frac{R}{2L}}$$

$$\left[L\left(-\frac{R}{2L} + \frac{R^2 t}{4L^2} - \frac{R}{2L}t\right) + R\left(-\frac{R}{2L}t + 1\right) + \frac{t}{C}\right] e^{-\frac{R}{2L}t}$$

$$\left[\underbrace{\left(-\frac{R}{2} - \frac{R}{2} + R\right)}_{0} + t\underbrace{\left(\frac{R^2}{4L} - \frac{R^2}{2L} + \frac{1}{C}\right)}_{0}\right] e^{-\frac{R}{2L}t}$$

so in the third case there are still two smoothly decaying solutions

$$Q(t) = e^{-\frac{R}{2L}t}(a+bt)$$

$$I(t) = -\frac{R}{2L}e^{-\frac{R}{2L}t}(a+bt) + be^{-\frac{R}{2L}t}$$

at $t=0$ $\quad Q(0) = a$

$$I(0) = -\frac{R}{2L}Q(0) + b$$

$$b = I(0) + \frac{R}{2L}Q(0)$$

$$\boxed{\begin{aligned} Q(t) &= e^{-\frac{R}{2L}t}\left(Q(0) + \left(I(0)+\frac{R}{2L}Q(0)\right)t\right) \\ I(t) &= e^{-\frac{R}{2L}t}\left(-\frac{Q(0)R}{2L} + \left(I(0)+\frac{R}{2L}Q(0)\right)\left(1-\frac{R}{2L}t\right)\right) \end{aligned}}$$

It is easy to check that these have the correct initial conditions.

These equations are analogous to the equations for a harmonic oscillator in a viscous fluid - where R represents the viscosity.

AC circuits

AC circuits have an oscillating EMF

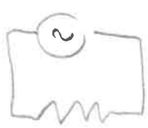
we use the symbol

$$\sim \; = \; \mathcal{E}_0 \cos(\omega t)$$

$$\mathcal{E}_0 \sin(\omega t)$$

$$\mathcal{E}_0 e^{i\omega t}$$

these are all possible - consider



$$\mathcal{E} - IR = 0$$

in this case

$$IR = \mathcal{E}_0 e^{i\omega t}$$

$$I = \frac{\mathcal{E}_0}{R} e^{i\omega t} \qquad (e^{i\omega t} = \cos\omega t + i \sin\omega t)$$

the real or imaginary of this or

$$I_r = \frac{\mathcal{E}_0}{R} \cos(\omega t)$$

$$I_i = \frac{\mathcal{E}_0}{R} \sin(\omega t)$$

$$\mathcal{E} - L \frac{dI}{dt}$$

$$L \frac{dI}{dt} = \mathcal{E}_0 e^{i\omega t}$$

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L} e^{i\omega t}$$

integrating

$$\int^{t} \frac{dI}{dt} = \int \frac{\mathcal{E}_0}{L} e^{i\omega t} dt$$

$$I(t) - I(0) = \frac{1}{i\omega}\frac{\mathcal{E}_0}{L} e^{i\omega t} - \frac{1}{i\omega}\frac{\mathcal{E}_0}{L}$$

note $i^{-1} = e^{-i\frac{\pi}{2}}$

$$I(t) - I(0) = \frac{\mathcal{E}}{L\omega}\left(e^{i(\omega t - \frac{\pi}{2})} - e^{-i\frac{\pi}{2}}\right)$$

for the current

$$I(t) = \frac{\mathcal{E}}{L\omega} e^{i(\omega t - \frac{\pi}{2})}$$

taking real a imaginary part

$$I_r(t) = \frac{\mathcal{E}}{L\omega} \cos(\omega t - \tfrac{\pi}{2})$$

$$I_i(t) = \frac{\mathcal{E}}{L\omega} \sin(\omega t - \tfrac{\pi}{2})$$

we define $X_L = L\omega$ called the inductive reactance

$$\boxed{\mathcal{E}(t) = X_L I(t)}$$

where the phase of the current in the inductor is $\frac{\pi}{2}$ behind the phase of the emf

For a capacitor

$$\mathcal{E} - \frac{Q}{C} = 0$$

$$\mathcal{E}_0 e^{i\omega t} = \frac{Q}{C}$$

the current in the capacitor

$$I_c = \frac{dQ}{dt} = C\mathcal{E}_0 i\omega e^{i\omega t}$$
$$= \mathcal{E}_0 C\omega e^{i(\omega t + \frac{\pi}{2})}$$

we define the capacitive reactance

$$X_c = \frac{1}{C\omega}$$

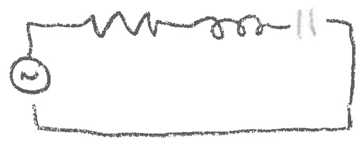
and we have

$$\mathcal{E}(t) = I(t) X_c$$

where the phase of the current through the resistor is $\frac{\pi}{2}$ ahead of the phase of the EMF.

$\therefore \quad \mathcal{E}(t) = I(t) R$		in phase
$= I(t) X_c$	$X_c = \frac{1}{\omega C}$	$+\frac{\pi}{2}$
$= I(t) X_L$	$X_L = \omega L$	$-\frac{\pi}{2}$

Next consider a general driven RLC circuit



$$\mathcal{E} = \mathcal{E}_0 \cos \omega t$$

The energy conservation equation gives

$$-L\frac{dI}{dt} - IR - \frac{Q}{C} + \mathcal{E}_0 \cos \omega t = 0$$

express this in terms of Q

$$I = \frac{dQ}{dt} \qquad \frac{dI}{dt} = \frac{d^2Q}{dt^2}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \frac{\mathcal{E}_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \qquad (1)$$

let Q_1 Q_2 Q_3 be solutions to

$$L\frac{d^2Q_1}{dt^2} + R\frac{dQ_1}{dt} + \frac{1}{C}Q_1 = 0$$

$$L\frac{d^2Q_2}{dt^2} + R\frac{dQ_2}{dt} + \frac{1}{C}Q_2 = \frac{\mathcal{E}_0}{2}e^{i\omega t}$$

$$L\frac{d^2Q_3}{dt^2} + R\frac{dQ_3}{dt} + \frac{1}{C}Q_3 = \frac{\mathcal{E}_0}{2}e^{-i\omega t}$$

If we define $Q = Q_1 + Q_2 + Q_3$ and add the 3 equations above we get a solution to (1)

The equation for Q_1 we just studied and found that the solutions had the form

$$Q_1(t) = Q_+ e^{a_+ t} + Q_- e^{a_- t}$$

where both eventually decayed to 0.

We can find solutions to Q_2 and Q_3 assuming they have the form $Q_2(\omega) e^{i\omega t}$ $Q_3(\omega) e^{-i\omega t}$. Using these in the equation gives

$$\left(-L\omega^2 + iR\omega + \frac{1}{C}\right) Q_2(\omega) e^{i\omega t} = \frac{\varepsilon}{2} e^{i\omega t}$$

$$\left(-L^2\omega^2 - iR\omega + \frac{1}{C}\right) Q_3(\omega) e^{-i\omega t} = \frac{\varepsilon}{2} e^{-i\omega t}$$

$$Q_2(t) = \frac{\varepsilon}{2} \frac{1}{-L\omega^2 + iR\omega + \frac{1}{C}} e^{i\omega t}$$

$$Q_3(t) = \frac{\varepsilon}{2} \frac{1}{-L\omega^2 - iR\omega + \frac{1}{C}} e^{-i\omega t}$$

factoring out $\pm i\omega$ from the denominator

$$Q_2(t) = \frac{\varepsilon}{2\omega} \frac{1}{iL\omega + R - \frac{i}{\omega C}} e^{i(\omega t - \frac{\pi}{2})}$$

$$Q_3(t) = \frac{\varepsilon}{2\omega} \frac{1}{-iL\omega + R + \frac{i}{\omega C}} e^{-i(\omega t - \frac{\pi}{2})}$$

we can express these in terms of X_c X_L

$$Q_2(t) = \frac{\mathcal{E}}{2\omega} \frac{1}{R+i(X_L-X_c)} e^{i(\omega t - \frac{\pi}{2})}$$

$$Q_3(t) = \frac{\mathcal{E}}{2\omega} \frac{1}{R-i(X_L-X_c)} e^{-i(\omega t - \frac{\pi}{2})}$$

multiply the denominator and numerator
$R \pm i(X_L-X_c)$

$$Q_2(t) = \frac{\mathcal{E}}{2\omega} \frac{R-i(X_L-X_c)}{R^2+(X_L-X_c)^2} e^{i(\omega t - \frac{\pi}{2})}$$

$$Q_3(t) = \frac{\mathcal{E}}{2\omega} \frac{R+i(X_L-X_c)}{R^2+(X_L-X_c)^2} e^{-i(\omega t - \frac{\pi}{2})}$$

we define the impedance of this circuit

$$Z = \sqrt{R^2+(X_L-X_c)^2} \qquad \frac{R-i(X_L-X_c)}{\sqrt{R^2+(X_L-X_c)^2}} = e^{i\phi}$$

$$Q_2(t) = \frac{\mathcal{E}}{2\omega} \frac{1}{Z} e^{i(\omega t - \frac{\pi}{2} + \phi)}$$

$$Q_3(t) = \frac{\mathcal{E}}{2\omega} \frac{1}{Z} e^{-i(\omega t - \frac{\pi}{2} + \phi)}$$

putting everything together.

$$Q(t) = Q_1(t) + Q_2(t) + Q_3(t) =$$

$$Q_+ e^{a_+ t} + Q_- e^{a_- t} + \frac{\mathcal{E}_0}{\omega Z} \frac{1}{2} \left(e^{i(\omega t - \frac{\pi}{2} + \phi)} - e^{-i(\omega t - \frac{\pi}{2} + \phi)} \right) =$$

$$Q_+ e^{a_+ t} + Q_- e^{a_- t} + \frac{\mathcal{E}_0}{\omega Z} \cos(\omega t - \frac{\pi}{2} + \phi)$$

In this expression Q_+ and Q_- are arbitrary. They can be used to fix $Q(0)$ and $I(0)$.

eventually $e^{a_+ t}$ $e^{a_- t} \to 0$ for large t and what remains is

$$Q(t) \to \frac{\mathcal{E}_0}{\omega Z} \frac{1}{2} \left(e^{i(\omega t - \frac{\pi}{2} + \phi)} + e^{-i(\omega t - \frac{\pi}{2} + \phi)}\right)$$

differentiating gives the current

$$I(t) = \frac{dQ}{dt} = -\frac{\mathcal{E}_0}{Z} \frac{1}{2i} \left(e^{i(\omega t - \frac{\pi}{2} + \phi)} - e^{-i(\omega t - \frac{\pi}{2} + \phi)}\right)$$

$$= -\frac{\mathcal{E}}{Z} \sin(\omega t - \tfrac{\pi}{2} + \phi)$$

$$Q(t) = \frac{\mathcal{E}_0}{\omega Z} \cos(\omega t - \tfrac{\pi}{2} + \phi)$$

note that the denominator will be smallest when

$X_L = X_C$ since $Z = \sqrt{R^2 + (X_L - X_C)}$

$L\omega = \frac{1}{\omega C}$ $\quad \omega^2 = \frac{1}{LC}$

when ω is the natural frequency of the LC circuit. - this means that the current is maximized when $\omega = 1/\sqrt{LC}$