Lecture 2

Coulombs Law

Superposition Principle

continuous charge distributions

Coulombs Law:

The force between 2 charged particles

① is proportional to each charge

② falls off like the square of the distance between the charges

③ acts along the line between charges; repulsive for like charges, attractive for unlike charges

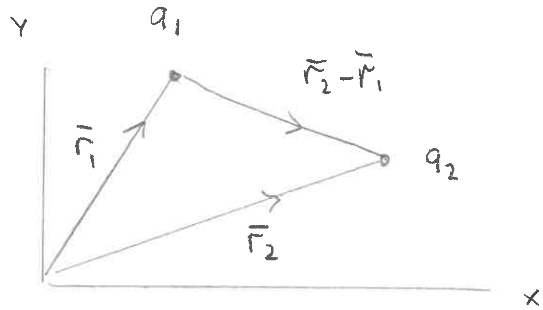
$\bar{F}_{12}$ = force on charge 1 due to the charge 2.

$$\bar{F}_{12} = k q_1 q_2 \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_2 - \bar{r}_1|^3}$$

$$k = 8.99 \times 10^{-9} \frac{N m^2}{C^2} = \frac{1}{4\pi\epsilon_0}$$

$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$

To evaluate we use vectors - choose a coordinate system



example -

$\bar{r}_1 = (x_1, y_1, z_1)$

$\bar{r}_2 = (x_2\ y_2\ z_2)$

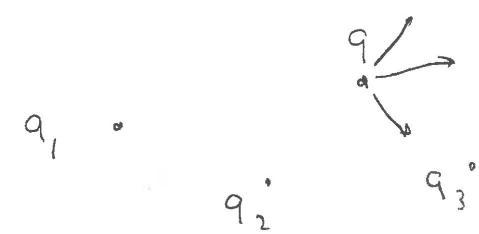
$$\bar{F}_{12} = \frac{k q_1 q_2}{((x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2)^{3/2}} (x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

what is the magnitude of the force between charges $q_1 = 1C$ $q_2 = -2C$ separated by 2 meters

$$\boxed{|\bar{F}| \approx \left| 9 \times 10^9 \frac{Nm^2}{C^2} \frac{(1C)(-2C)}{(2)(2)\,m^2} \right| = 4.5 \times 10^9\ N}$$

superposition principle

The force on one charge is the sum of the forces on due to all of the other charges

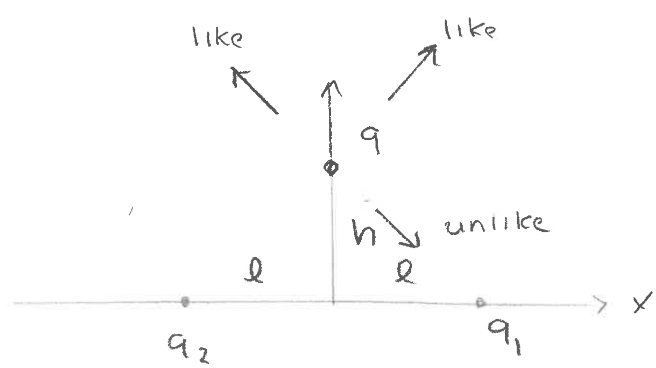


need to sum all vectors

$$\vec{F}_q = \bar{F}_{qq_1} + \bar{F}_{qq_2} + \cdots + \bar{F}_{qq_N}$$

$$= qk \sum_{n=1}^{N} q_n \frac{\bar{r} - \bar{r}_n}{|\bar{r} - \bar{r}_n|^3}$$

here the sum corresponds to vector addition



assume $q_1 = \pm q_2$

$\bar{r} = (0\ h)$ $\quad |\bar{r} - \bar{r}_1| = \sqrt{\ell^2 + h^2}$

$\bar{r}_1 = (-\ell\ 0)$ $\quad |\bar{r} - \bar{r}_2| = \sqrt{(-\ell)^2 + h^2}$

$\bar{r}_2 = (+\ell\ 0)$

$$\bar{F}_q = \bar{F}_{qq_1} + \bar{F}_{qq_2} =$$

$$kq\left(q_1 \frac{(h, \ell)}{(h^2+\ell^2)^{3/2}} + q_2 \frac{(h, -\ell)}{(h^2+\ell^2)^{3/2}}\right)$$

$q_1 = q_2$

$$kqq_1 \frac{(2h, 0)}{(h^2+\ell^2)^{3/2}}$$

$q_1 = -q_2$

$$kqq_1 \frac{(0, 2\ell)}{(h^2+h^2)^{3/2}}$$

comparison of forces on 2 protons at 10^{-15} m

$$|grav| = \left|\frac{G m_p^2}{r^2}\right| = \frac{(6.67\times10^{-11}\ \frac{Nm^2}{kg^2})(1.7\times10^{-27}\ kg)^2}{(10^{-15}m)^2} \approx 19.3\times10^{-35}\ N$$

$$|coulomb| = \left|\frac{kq_p^2}{r^2}\right| = (9\times10^9\ \frac{Nm^2}{C^2})\frac{(1.6\times10^{-19}\ C)^2}{(10^{-15}m)^2} \approx 2.3\times10^2\ N^2$$

$$|strong| = \left|\frac{g}{r^2}e^{-\mu r}\right| = 3.4\times10^{-26}\ Nm^2\ \frac{e^{-10^{-15}/1.5\times10^{-15}}}{(10^{-15}m)^2} \approx 1.7\times10^4\ N$$

$$\frac{F_s}{F_e} \approx 60.1 \ @\ 10^{-15}\ m$$

continuous charge distributions

While charge comes in integer multiples of the electron or proton charge, for macroscopic physics a Coulomb has 6.24×10^{18} charges - it is impractical to treat them as discrete. It is useful to treat them as approximately continuous

① line charges

$\lambda = \frac{Q}{\ell}$ = charge per unit length

$\lambda \, d\ell$ = charge in length $d\ell$.

② surface charges

$\sigma = \frac{Q}{A}$ = charge per unit area

$\sigma \, dA$ = charge in area dA

③ volume charge

$\rho = \frac{Q}{V}$

$\rho \, dV$ = charge in volume dV.

In general $\lambda \to \lambda(\bar{r})$ $\sigma \to \sigma(\bar{r})$ $\rho \to \rho(\bar{r})$

these charge densities could depend on position

in each case

$$\bar{F}_{q\,line}(\bar{r}) = kq \sum_{n=1}^{N} \frac{q_n(\bar{r}-\bar{r}_n)}{|r-r_n|^3}$$

$$\simeq kq \int_0^{\ell} \lambda(\bar{r}(s)) \frac{\bar{r}-\bar{r}(s)}{|\bar{r}-\bar{r}(s)|^3}\, ds$$

$$\bar{F}_{q\,surface}(\bar{r}) = kq \sum_{n=1}^{N} \frac{q_n(\bar{r}-\bar{r}_n)}{|r-r_n|^3} \simeq$$

$$\simeq kq \int dA\, \sigma(\bar{r}(A)) \frac{\bar{r}-\bar{r}(A)}{|r-\bar{r}(A)|^3}$$

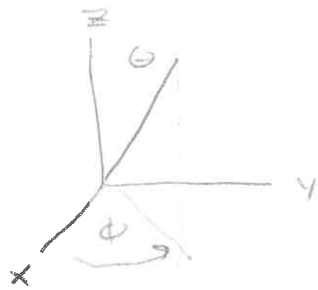
$$\bar{F}_{q\,volume}(\bar{r}) \simeq kq \int dV\, \rho(r(V)) \frac{\bar{r}-\bar{r}(A)}{|\bar{r}-\bar{r}(A)|^3}$$

In all cases $\sum q_n \to \int \lambda\, d\ell\,; \int \sigma\, dA\,; \int \rho\, dV$

Force on a charged sphere with spherically symmetric charge distribution. Assume sphere has radius R.

$$\bar{\rho} = \rho(r)$$

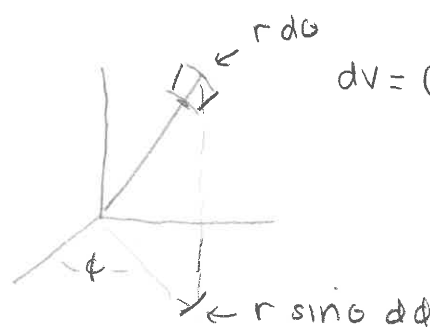
We assume that the sphere is centered at the origin of the coordinate system and use spherical coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

spherically symmetric means $\rho(r)$ is independent of θ, ϕ.

$\theta: 0 \rightarrow \pi \qquad \phi: 0 \rightarrow 2\pi$



$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$$

since we are free to choose coordinate axes - let the z axis be along the line between the center of the sphere and the particle

case 1 $r = (0\,0\,z) > R$

$$\bar{F} = kq \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{R} r^2 dr\, \rho(r)\, \times$$

$$\frac{(-r\sin\theta\cos\phi,\ -r\sin\theta\sin\phi,\ z - r\cos\theta)}{\underbrace{(r^2\sin^2\theta + (z-r\cos\theta)^2)}_{(r^2 - 2rz\cos\theta + z^2)}{}^{3/2}}$$

we note $\int_0^{2\pi} \cos\phi\, d\phi = \int_0^{2\pi} \sin\phi\, d\phi = 0$

$\sin 2\pi - \sin 0 = 0 \qquad -\cos 2\pi + \cos 0 = 0$

this means that the force is in the z direction

$$\bar{F} = kq \cdot 2\pi \int_0^{R} r^2 \rho(r)\, dr \int_0^{\pi} \frac{(z - r\cos\theta)}{(r^2 - 2rz\cos\theta + z^2)^{3/2}} \sin\theta\, d\theta$$

note $\frac{d}{dz} \frac{1}{(r^2 - 2rz\cos\theta + z^2)^{\frac{1}{2}}} = -\frac{1}{2} \frac{-2r\cos\theta + 2z}{(r^2 - 2rz\cos\theta + z^2)^{3/2}}$

$$= -\frac{z - r\cos\theta}{(r^2 - 2rz\cos\theta + z^2)^{3/2}}$$

$$2\pi k q \int_0^R r^2 \rho(r)\,dr \,\frac{d}{dz}\int_0^{\pi} \frac{-1}{\sqrt{r^2+z^2-2rz\cos\theta}} \sin\theta\, d\theta$$

let $\quad u = r^2+z^2-2rz\cos\theta \qquad du = 2rz\sin\theta\, d\theta$

$$2\pi k q \int_0^R r^2 \rho(r)\,dr \,\frac{d}{dz}\int_{(z-r)^2}^{(z+r)^2} -\frac{du}{u^{1/2}}\cdot\frac{1}{2rz}$$

$$2\pi k q \int_0^R r^2 \rho(r)\,dr \left(\frac{-1}{2r}\right)\frac{d}{dz}\underbrace{\left\{2(z+r)-2(z-r)\right\}\frac{1}{z}}_{\frac{4r}{z}}$$

$$\frac{4\pi k q}{z^2}\int r^2 \rho(r)\,dr$$

but for a spherical distribution

$$Q = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^R r^2 \rho(r)\,dr =$$

$$= 2\pi\,(-\cos\pi + \cos 0)\int_0^R r^2 \rho(r)\,dr$$

$$= 4\pi \int_0^R r^2 \rho(r)\,dr$$

$$\therefore \quad \boxed{\bar{F} = \frac{kQq}{z^2}\,\hat{z}}$$

This shows that for $z > R$ the force on q is identical to the force on q due to a charge Q at the origin what happens if $z < R$ –

everything else is the same

$$\bar{F} = (0,0,F_z)$$

$$F_z = 2\pi k q \int_0^R \rho(r) r^2 dr \frac{d}{dz} \int_{(z-r)^2}^{(z+r)^2} -\frac{1}{2rz} u^{-1/2} =$$

$$2\pi k q \int_0^z \rho(r) r^2 dr \frac{d}{dz} \int_{(z-r)^2}^{(z+r)^2} -\frac{1}{2rz} u^{-1/2} + \qquad r < z$$

$$2\pi k q \int_z^R \rho(r) r^2 dr \frac{d}{dz} \int_{(z-r)^2}^{(r+r)^2} -\frac{1}{2rz} u^{-1/2} \qquad r > z$$

$$2\pi k q \int_0^z \rho(r) r^2 dr \frac{d}{dz} \Big(\underbrace{\frac{2}{2rz} (z+r - z+r)}_{\frac{2}{z}\ 2} \Big) +$$

$$2\pi k q \int_0^z \rho(r) r^2 dr \frac{d}{dz} \Big(\underbrace{\frac{2}{2rz} (z+r - r+z)}_{\left(\frac{2}{r}\right) \to 0} \Big)$$

the second term vanishes; the first has the form

$$F_z = kq \frac{Q(z)}{z^2}$$

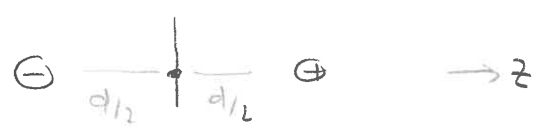
where $Q(z)$ is the charge in the part of the sphere with radius less than z.

Note that the radial dependence of the charge distribution does not matter.

If the sphere is hollow, there is no charge for $z <$ inner radius so there is no force on a particle inside a hollow spherically symmetric charge distribution.

Force due to a dipole

A dipole is a pair of opposite charges separated by a distance "d"



We choose coordinates so the charges are on the z axis dipoles are interesting because it is what you see far from a neutral charge distribution

$$\bar{F}(\bar{r}) = kqq_d \left(\frac{\bar{r} - d\frac{\hat{z}}{2}}{|\bar{r} - \frac{d\hat{z}}{2}|^3} - \frac{\bar{r} + d\frac{\hat{z}}{2}}{|\bar{r} + \frac{d\hat{z}}{2}|^3} \right)$$

note

$$|\bar{r} \pm \tfrac{d}{2}\hat{z}| = \left| r^2 \pm d\,\bar{r}\cdot\hat{z} + \frac{d^2}{4} \right|^{1/2}$$

$$\bar{\nabla} \frac{1}{|\bar{r} \pm \frac{d}{2}\hat{z}|} = \bar{\nabla} \frac{1}{\sqrt{r^2 \pm d\,\bar{r}\cdot\hat{z} + \frac{d^2}{4}}} =$$

$$-\frac{1}{2} \frac{1}{(r^2 \pm d\,\bar{r}\cdot\hat{z} + \frac{d^2}{4})^3} \left(2\bar{r} \pm d\hat{z} \right) =$$

$$- \frac{\bar{r} \pm \frac{d}{2}\hat{z}}{|\bar{r} \pm \frac{d\hat{z}}{2}|^3}$$

$$\therefore \quad \bar{F}(\bar{r}) = kqq_d(-\bar{\nabla}) \left\{ \frac{1}{\sqrt{r^2 - d\,\bar{r}\cdot\hat{z} + \frac{d^2}{4}}} - \frac{1}{\sqrt{r^2 + \bar{r}\cdot\hat{z}d + \frac{d^2}{4}}} \right\}$$

expanding

$$\frac{1}{\sqrt{r^2 \pm \bar{r}\cdot\hat{z}d + \frac{d^2}{4}}} = \frac{1}{r} \frac{1}{\sqrt{1 \pm \frac{\hat{r}\cdot\hat{z}}{r} + \frac{d^2}{4r^2}}} \simeq \frac{1}{r}\left(1 - \frac{1}{2}\left(\pm \frac{\hat{r}\cdot\hat{z}}{r}\right) + \cdots \right)$$

$$\simeq \frac{1}{r} \mp \frac{\hat{r}\cdot\hat{z}}{2r^2} + \cdots$$

using this in the equation on the top of the last page

$$F(\bar{r}) \simeq kqq_d\,(-\bar{\nabla})\left(\frac{1}{r} + d\frac{\hat{r}\cdot\hat{z}}{2r^2} + \cdots - \frac{1}{r} + d\frac{\hat{r}\cdot\hat{z}}{2r^2} + \cdots\right)$$

$$\simeq kq(q_D d)(-\bar{\nabla})\,\frac{\hat{r}\cdot\hat{z}}{r^2} + \cdots$$

the small term fall off like higher powers of $\left(\frac{d}{r}\right)$. In the limit of large r or $(q_D d) = \text{const}$ $q_D \to \infty$ $d \to 0$ gives

$$\bar{F}(r) \to kq(q_D d)\left(-\bar{\nabla}\left(\frac{\hat{r}\cdot\hat{z}}{r^2}\right)\right)$$

$q_D d\,\hat{z} = \bar{p}$ is called the dipole moment

$$\bar{F}(r) = kq\left(-\bar{\nabla}\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right)\right)$$

$$\sim kq\left(\frac{3\,\bar{r}(\bar{r}\cdot\bar{p})}{r^5} - \frac{\bar{p}(r^2)}{r^5}\right)$$