Lecture 29

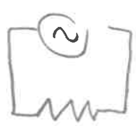
Driven RLC circuits

$\mathcal{E}_0 \cos(\omega t)$ or

$\mathcal{E}_0 \sin(\omega t)$ or

$\mathcal{E}_0 e^{\pm i\omega t}$

single elements



$\mathcal{E} = IR$

$I = \frac{\mathcal{E}_0}{R} e^{i\omega t}$

$\mathcal{E} = \frac{Q}{C}$

$Q(t) = C\mathcal{E}_0 e^{i\omega t}$

$I(t) = \frac{dQ}{dt} = \omega C \mathcal{E}_0 i e^{i\omega t}$

$i = e^{i\frac{\pi}{2}}$

$I(t) = \omega C \mathcal{E}_0 e^{i(\omega t + \frac{\pi}{2})}$

$X_c = \frac{1}{\omega C}$ capacitive reactance

current through capacitor is 90° ahead of current in EMF.

$$\mathcal{E} = L\frac{dI}{dt}$$

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L} e^{i\omega t}$$

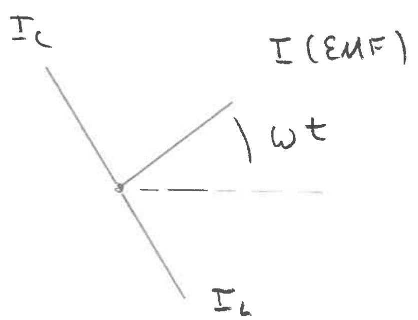
$$I(t) = \frac{\mathcal{E}_0}{L\omega}\frac{1}{i} e^{i\omega t}$$

$$\frac{1}{i} = -i = e^{-i\frac{\pi}{2}}$$

$$I(t) = \frac{\mathcal{E}}{L\omega} e^{i(\omega t - \frac{\pi}{2})}$$

$X_L = L\omega$ (inductive reactance)

current through inductor is 90° behind current through EMF



RLC Circuits

Driven RLC circuits

$$\odot = \mathcal{E}_0 \cos(\omega t)$$

The loop equation gives

$$\mathcal{E}_0 \cos(\omega t) - L\frac{dI}{dt} - RI - \frac{1}{C}Q = 0$$

using

$$I = \frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}_0 \cos(\omega t)$$

To solve this write

$$\mathcal{E}_0 \cos(\omega t) = \frac{\mathcal{E}_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{\mathcal{E}_0}{2L}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

We let $Q(t)$ be a sum of 3 terms

$$Q = Q_1(t) + Q_2(t) + Q_3(t)$$

where

$$\frac{d^2Q_1}{dt^2} + \frac{R}{L}\frac{dQ_1}{dt} + \frac{1}{LC}Q_1(t) = 0 \qquad (1)$$

$$\frac{d^2Q_2}{dt^2} + \frac{R}{L}\frac{dQ_2}{dt} + \frac{1}{LC}Q_2 = \frac{\mathcal{E}_0}{2L} e^{i\omega t} \qquad (2)$$

$$\frac{d^2Q_3}{dt^2} + \frac{R}{L}\frac{dQ_3}{dt} + \frac{1}{LC}Q_3 = \frac{\mathcal{E}_0}{2L} e^{-i\omega t} \qquad (3)$$

equation (1) is called the homogeneous equation. If we assume solutions of the form e^{at} we find

$$a^2 + \frac{R}{L}a + \frac{1}{LC} = 0$$

$$a_\pm = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

with solutions $e^{a_\pm t}$

$$\boxed{Q(t) = C_+ e^{a_+ t} + C_- e^{a_- t}}$$

where C_+ and C_- are arbitrary coefficients. These are used to fix the initial charge on the capacitor and current.

We need specific solutions to the other 2 equations. We look for soltions of the form $A_\pm e^{\pm i\omega t}$

using these solutions in the Q_2 and Q_3 equations gives

$$A_+ \left(-\omega^2 + i\omega \frac{R}{L} + \frac{1}{LC}\right) e^{i\omega t} = \frac{\mathcal{E}_0}{2L} e^{i\omega t}$$

$$A_- \left(-\omega^2 - i\omega \frac{R}{L} + \frac{1}{LC}\right) e^{-i\omega t} = \frac{\mathcal{E}_0}{2L} e^{-i\omega t}$$

these can be solved $A_\pm$

$$A_\pm = \frac{\mathcal{E}_0}{2L} \frac{1}{-\omega^2 \pm i\omega \frac{R}{L} + \frac{1}{LC}} =$$

$$= \frac{\mathcal{E}_0}{2\omega} \frac{1}{-L\omega \mp iR + \frac{1}{\omega C}} =$$

$$= \frac{\mathcal{E}}{2\omega} \frac{1}{(X_C - X_L) \pm iR} =$$

where $X_L = L\omega$ $X_C = \frac{1}{\omega C}$ we write the solutions as

$$Q_2 = \frac{1}{\omega} \frac{1}{(X_C - X_L) + iR} \frac{\mathcal{E}_0}{2} e^{i\omega t}$$

$$Q_3 = \frac{1}{\omega} \frac{1}{(X_C - X_L) - iR} \frac{\mathcal{E}_0}{2} e^{-i\omega t}$$

the full solution is

$$Q_1 + Q_2 + Q_3 = Q =$$

$$C_+ e^{a_+ t} + C_- e^{a_- t} + \frac{1}{\omega} \frac{1}{(X_C - X_L) + iR} \frac{\mathcal{E}}{2} e^{i\omega t} + \frac{1}{\omega} \frac{1}{(X_C - X_L) - iR} \frac{\mathcal{E}}{2} e^{-i\omega t}$$

$$C_+ e^{a_+ t} + C_- e^{a_- t} + \frac{1}{i\omega} \frac{1}{R + i(X_L - X_C)} \frac{\mathcal{E}}{2} e^{i\omega t} + \frac{1}{-i\omega} \frac{1}{R - i(X_L - X_C)} \frac{\mathcal{E}}{2} e^{-i\omega t}$$

The current is the more useful quantity

$$I = \frac{dQ}{dt} = \frac{dQ_1}{dt} + \frac{dQ_2}{dt} + \frac{dQ_3}{dt}$$

$$= C_+ a_+ e^{a_+ t} + C_- a_- e^{c_- t} +$$

$$\frac{1}{R + i(X_L - X_C)} \frac{\mathcal{E}}{2} e^{i\omega t} + \frac{1}{R - i(X_L - X_C)} \frac{\mathcal{E}}{2} e^{-i\omega t}$$

we write

$$\frac{1}{R \pm i(X_L - X_C)} = \frac{1}{\sqrt{R^2 + (X_L - X_C)^2}} \frac{R \mp i(X_L - X_C)}{\sqrt{R^2 + (X_L - X_C)^2}}$$

The quantity

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$ is called the impedance of the circuit

$$\frac{R \mp i(X_L - X_C)}{\sqrt{R^2 + (X_L - X_C)^2}} = e^{i\phi} \qquad \phi = \text{phase constant}$$

with these definitions

$$\boxed{I(t) = C_+ a_+ e^{a_+ t} + C_- a_- e^{c_- t} + \frac{\mathcal{E}}{Z} \cos(\omega t - \phi)}$$

the solutions of the homogeneous equation can be used to fix $Q(0)$ $I(0)$. Independent of how they are chosen, because $a^{a_\pm t}$ eventually $\to 0$ for large $t \gg \frac{2L}{R}$ what remains is

$$I(t) \to \frac{\mathcal{E}_0}{Z} \cos(\omega t - \phi)$$

$$\mathcal{E} = \mathcal{E}_0 \cos(\omega t)$$

note that we could have started with $\frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})$

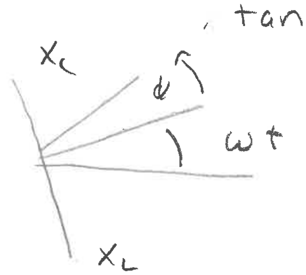
then we would get

$$I(t) \to \frac{\mathcal{E}_0}{Z} \sin(\omega t - \phi)$$

$$\mathcal{E} = \mathcal{E}_0 \sin(\omega t)$$

geometrically

, tan $\frac{X_L - X_C}{}$



we check the limiting values

if $R = L = 0$

$$\frac{-i X_C}{X_C} = e^{-i\frac{\pi}{2}} \Rightarrow \phi = -\frac{\pi}{2}$$

$$I(t) = \frac{\mathcal{E}}{X_C} \cos(\omega t + \tfrac{\pi}{2})$$

if $R = C = 0$

$$\frac{i X_L}{X_L} = e^{i\frac{\pi}{2}} = e^{i\phi} \qquad \phi = \frac{\pi}{2}$$

$$I(t) = \frac{\mathcal{E}}{X_L} \cos(\omega t + \tfrac{\pi}{2})$$

For RLC circuits the power is

$$P = I_2 R = \frac{\mathcal{E}^2}{Z^2} \cos^2(\omega t + \phi) R$$

what is of interest is the average power

$$\langle P \rangle = \frac{1}{T} \int_0^T \frac{\mathcal{E}^2}{Z^2} \cos^2(\omega t + \phi) R \, dt$$

$$= \frac{1}{2} \frac{\mathcal{E}^2 R}{Z^2} = \frac{1}{2} \frac{\mathcal{E}^2}{Z} \left(\frac{R}{Z}\right) = \frac{1}{2} \frac{\mathcal{E}^2}{Z} \cos\phi.$$

where

we can express this in term of the root mean square current and EMF

$$I_{Rms} = \left[\frac{1}{T} \int_0^T \left(\frac{\mathcal{E}}{Z} \cos(\omega t - \phi) \right)^2 dt \right]^{\frac{1}{2}} = \frac{\mathcal{E}}{\sqrt{2}\, Z}$$

$$\mathcal{E}_{Rms} = \left(\int_0^T \frac{1}{T} (\mathcal{E} \cos \omega t)^2 dt \right) = \frac{\mathcal{E}}{\sqrt{2}}$$

$$\boxed{I_{Rms} Z = \mathcal{E}_{rms}}$$

$$\boxed{\langle P \rangle = (I_{rms})(\mathcal{E}_{rms}) \cos \phi}$$

If the phase is near $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ there is no power through the resistor. When $\cos \phi = 0$ or π the power is maximal

Note that the impedance plays the role of resistance in AC circuits - when $\phi = \pm \frac{\pi}{2}$ the current and EMF are out of phase $\sim \sin\phi \cos\phi = \frac{1}{2} \sin 2\phi$ which will time average to 0.

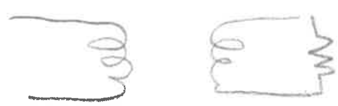
similarly if $N_1 > N_2$ $\mathcal{E}_2 < \mathcal{E}_1$, so depending on the ratio of turns of wire a transformer can change the EMF of an ac circuit

In an ideal transformer energy is not lost

$$P = I_1 V_1 = I_2 V_2$$

$$I_1 = I_2 \frac{V_2}{V_1} = I_2 \frac{N_2}{N_1}$$

this means that if the voltage goes up the current must go down



$$V_2 = I_2 R_2$$

$$V_2 = \frac{N_2}{N_1} J_1 \qquad I_2 = \frac{N_1}{N_2} I_1$$

$$R_1 = \frac{V_1}{I_1} = \frac{N_1}{N_2} V_2 \left(\frac{N_1}{N_2} \frac{1}{I_2}\right)$$

$$= \left(\frac{N_1}{N_2}\right)^2 R_2$$

This is the resistance seen on the other side of the transformer.

all of this assumes that $\phi = 0$

note that

$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

Z will be smallest when ω is chosen

so $X_L = X_C$

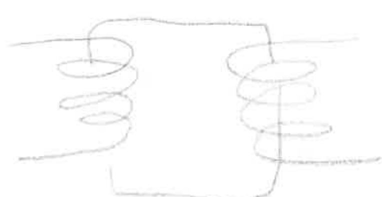
$$\omega L = \frac{1}{\omega C} \Rightarrow \omega^2 = \frac{1}{LC}$$

in that case

$$e^{i\phi} = 1 \Rightarrow \phi = 0$$

this is the condition that is needed to maximize power.

Transformers



because AC current is not steady state

$$\Phi = BA \qquad B = n\mu_0 I = n\mu_0 \frac{\varepsilon}{Z}\cos(\omega t + \phi)$$

for a solenoid with N

$$\frac{d\Phi}{dt} = \frac{dB}{dt} A = -A \cdot \mu_0 n \frac{\varepsilon}{Z} \omega \sin(\omega t - \phi)$$

The ac current results in an ac EMF by faraday's law

A transformer traps most of the flux in an iron core it goes into another set of coils were the change in flux creates an EMF

Ideally the field is the same in both sets of coils

$$\Phi_1 = N_1 BA \quad \Phi_2 = N_2 BA$$

$$\frac{\Phi_1}{N_1} = \frac{\Phi_2}{N_2}$$

$$\Phi_1 = \frac{N_1}{N_2}\Phi_2$$

$$\frac{d\Phi_1}{dt} = \frac{N_1}{N_2}\frac{d\Phi_2}{dt}$$

changing the magnetic field on one sides

$$\boxed{-\mathcal{E}_1 = -\frac{N_1}{N_2}\mathcal{E}_2}$$

If we put a time varying EMF in side 1 we see a time varying EMF in side 2 - but if $N_2 > N_1$ $\mathcal{E}_2 > \mathcal{E}_1$

This means that we can use a transformer to change resistance

to $R = R_1 + R_2$

$$I = \frac{\mathcal{E}}{R_1 + R_2}$$

the power is R_2 is

$$P_2 = I^2 R_2 = \frac{\mathcal{E}^2}{(R_1+R_2)^2} R_2$$

This is maximal when

$$\frac{dP_2}{dR_2} = \left[\frac{-2\mathcal{E}^2 R_2}{(R_1+R_2)^3} + \frac{\mathcal{E}^2}{(R_1+R_2)^2} \right] = 0$$

$$\frac{-2R_2 + R_1 + R_2}{(R_1+R_2)^3} = 0$$

or $R_1 = R_2$

So in order to get the most power out of R_1 we want to adjust R_2 using a transformer so $R_1 = R_2$

This is called impedance matching

Another important use of transformers in in power lines - If the line has resistance R - $P = I_{rms}^2 R$

so for fixed R the power lost is reduced by using a high voltage / low current