Lecture 31

AC circuits

Driven RLC circuits

For $EMF = \mathcal{E}_0 e^{i\omega t}$

$$Q(t) = C_+ e^{a_+ t} + C_- e^{a_- t} + \frac{1}{i\omega Z} \frac{R - i(X_L - X_C)}{Z} e^{i\omega t}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad X_L = L\omega \quad X_C = \frac{1}{C\omega}$$

$$\tan\phi = \frac{(X_L - X_C)}{Z}$$

$$Q(t) = C_+ e^{a_+ t} + C_- e^{a_- t} + \frac{\mathcal{E}_0}{i\omega Z} e^{i(\omega t - \phi)}$$

$$I(t) = \frac{dQ}{dt} = C_+ a_+ e^{a_+ t} + C_- a_- e^{a_- t} + \frac{\mathcal{E}_0}{Z} e^{i(\omega t - \phi)}$$

① The terms $e^{a_+ t}$, $e^{a_- t}$ which are solutions of the homogeneous equation fix the initial conditions but eventually become irrelevant because they both $\to 0$ exponentially as $t \to \infty$

② taking the real and imaginary parts of the solution gives real solutions (here I ignore the solutions to the homogeneous equation.)

$$Q(t) = \frac{\mathcal{E}_0}{Z\omega} \cos\left(\omega t - \phi - \frac{\pi}{2}\right) \qquad \mathcal{E} = \mathcal{E}_0 \cos(\omega t)$$

$$Q(t) = \frac{\mathcal{E}_0}{Z\omega} \sin\left(\omega t - \phi - \frac{\pi}{2}\right) \qquad \mathcal{E} = \mathcal{E}_0 \sin(\omega t)$$

for the currents I_c

$$I(t) = \frac{\mathcal{E}_0}{Z} \cos(\omega t - \phi) \qquad \mathcal{E} = \mathcal{E}_0 \cos(\omega t)$$

$$I(t) = \frac{\mathcal{E}_0}{Z} \sin(\omega t - \phi) \qquad \mathcal{E} = \mathcal{E}_0 \sin(\omega t)$$

③ resonance current through the resistor will be maximal when Z is minimal

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$\therefore$ when $X_L = X_C$ or $\omega^2 = \frac{1}{LC}$

special cases

$$\tan \phi = \frac{X_L - X_C}{R}$$

$$\cos \phi = \frac{R}{Z}$$

$$\sin \phi = \frac{X_L - X_C}{Z}$$

$X_L = R = 0$

$$\sin \phi = -1$$
$$\cos \phi = 0$$
$$\phi = -\pi/2$$

$X_C = R = 0$

$$\sin \phi = 1$$
$$\cos \phi = 0$$
$$\phi = \frac{\pi}{2}$$

$X_L = X_C = 0$

$$\cos \phi = 1$$
$$\sin \phi = 0$$
$$\phi = 0$$

these agree with our previous definitions

Power in resistor

$$P = I^2 R$$

$$= \left(\frac{\mathcal{E}_0}{Z}\right)^2 \cos^2(\omega t - \phi) R$$

If we average this over an period

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\langle P \rangle = \left(\frac{\mathcal{E}_0}{Z}\right)^2 \underbrace{\frac{1}{T}\int_0^T \cos^2(\omega t - \phi)\,dt}_{1/2} \cdot R$$

$$= \frac{\mathcal{E}_0^2}{2Z} \cdot \frac{R}{Z}$$

where $R/Z = \cos\phi$

$$\boxed{\langle P \rangle = \frac{\mathcal{E}_0^2}{2Z} \cos(\phi)}$$

To maximize the power through the resistor we want $\phi = 0$

this is also the condition that minimizes Z $\left(X_L = X_C \Rightarrow \omega = 1/\sqrt{LC}\right)$

while the power changes periodically, it is always positive, this is not true for the EMF and current. what is normally measured is the root mean square current or emf

$$\left[\frac{1}{T}\int_0^T I^2(t)\,dt\right]^{1/2} = I_{rms} = \left[\left(\frac{\mathcal{E}_0}{Z}\right)^2 \frac{1}{T}\int_0^T \cos^2(\omega t-\phi)\,dt\right]^{1/2}$$

$$I_{rms} = \frac{1}{\sqrt{2}}\frac{\mathcal{E}_0}{Z}$$

similarly

$$\left[\frac{1}{T}\int_0^t \mathcal{E}^2(t)\,dt\right] = \mathcal{E}_{rms} = \left[\mathcal{E}_0^2 \frac{1}{T} \cos(\omega t-\phi)\right]^{1/2}$$

$$\mathcal{E}_{rms} = \frac{1}{\sqrt{2}}\mathcal{E}_0$$

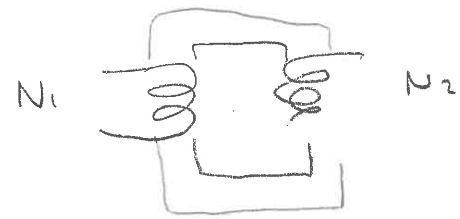
It follows that

$$\langle P\rangle = I_{rms}^2 R \cos\phi = \frac{1}{2}\frac{\mathcal{E}_0^2}{Z}\cos\phi$$

$$= \mathcal{E}_{rms} I_{rms} \cos\phi = \frac{1}{2}\mathcal{E}_0 I_0 \cos\phi$$

this mean the average power has DC looking formulas if we replace currents and EMF's by <u>RMS</u> currents and EMF

Ideal transformers - these are devices based on the idea of mutual inductance



A transformer has N_1 turns of wire wrapped around one side of and iron ring and N_2 on the other side - In an ideal transformer the magnetic flux is trapped in the ring so both coils feel the same field

$$\Phi_1 = N_1 B A \qquad \Phi_2 = N_2 B A$$

$$= L_1 I_1 \qquad = L_2 I_2$$

$$\mathcal{E}_1 = -\frac{d\Phi}{dt} = -N_1 A_2 \frac{dB}{dt}$$

changing EMF in 1 $\Rightarrow \frac{dB}{dt} \rightarrow$ the same $\frac{dB}{dt}$ results in a change in flux in coil 2

$$\mathcal{E}_2 = -N_2 A_2 \frac{dB}{dt}$$

If both coils have the same area

$$\frac{\mathcal{E}_2}{N_2} = \frac{\mathcal{E}_1}{N_1}$$

or

$$\boxed{\mathcal{E}_2 = \frac{N_2}{N_1}\mathcal{E}_1}$$

This means that a transformer can be use change the EMF

$$5V = \frac{N_2}{N_1} 110V$$

$$\frac{N_1}{N_2} = \frac{110}{5} = 22$$

with this ratio 22 we can step down from 110, like you would in a cell phone changer

since the frequency does not change we also have

$$\boxed{\mathcal{E}_2^{RMS} = \frac{N_2}{N_1}\mathcal{E}_1^{R}}$$

what resistance does the EMF feel

energy is conserve in an ideal transform

$$\frac{\mathcal{E}_1^2}{R_1} = \frac{\mathcal{E}_2^2}{R_2}$$

$$\frac{R_1}{R_2} = \frac{\mathcal{E}_1^2}{\mathcal{E}_2^2} = \frac{N_1^2}{N_2^2}$$

$$\boxed{R_1 = \left(\frac{N_1}{N_2}\right)^2 R_2}$$

we can use this to find the relation between the RMS current

$$I^1_{rms}\,\mathcal{E}^1_{rms} = I^2_{rms}\,\mathcal{E}^2_{rms}$$

$$\boxed{\frac{I^1_{RMS}}{I_{2RM}} = \frac{\mathcal{E}_{2\,RMS}}{\mathcal{E}_{1\,RM}} = \frac{N_2}{N_1}}$$

uses of transformers

efficient power distribution

power loss in a resistor is $I_{rms}^2 R$ - increasing the voltage by a factor of 100 reduces the current by a factor of 100 - thus reduces the power loss by a factor of $(100)^2$

Another use of transformer is impedance matching

* tune both circuits so $\phi = 0$,

R_2 is the resistance in my resistor - to maximize the power in R_2

$$P = I^2 R_2 = \left(\frac{\mathcal{E}}{R_1+R_2}\right)^2 R_2$$

$$\frac{dP}{dR_2} = \left(\frac{\mathcal{E}}{R_1+R_2}\right)^2 - \frac{2\mathcal{E}^2}{(R_1+R_2)^3} R_2 = 0$$

$$1 = \frac{2R_2}{R_1+R_2} \qquad R_1+R_2 = 2R_2 \qquad R_1 = R_2$$

so if we want to maximize the transmitted power - we can use a transformer to change the effective resistance to maximized power.

This is an important consideration in loudspeakers. It is called impedance matching

Maxwells Equations

① Coulombs Law $\quad \bar{F} = q\bar{E}$

$$E(\bar{r}) = \frac{1}{4\pi\epsilon_0} \sum q_i \frac{\bar{r} - \bar{r}_i}{|\bar{r} - \bar{r}_i|^3}$$

$$\oint \bar{E} \cdot \hat{n}\, dA = \frac{q}{4\pi\epsilon_0} \qquad \text{(Gauss Law)}$$

$$\bar{E} = -\nabla V \qquad (\bar{\nabla} \times \bar{E} = 0)$$

② Biot Savart law

$$dB(\bar{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\bar{l}' \times (\bar{r} - \bar{r}')}{|r - r'|^3}$$

$$\oint \bar{B}(r) \cdot d\bar{l} = \frac{\mu_0}{I} \qquad \text{(Amperes Law)}$$

③ Faradays Law

$$\frac{d\Phi}{dt} = -\mathcal{E}$$

$$\frac{d}{dt}\int \bar{B}\cdot\hat{n}\,dA = -\oint \bar{E}\cdot d\ell$$

$$\int\left(\frac{\partial B}{\partial t}\cdot\hat{n}\,dA\right) = -\int(\bar{\nabla}\times\bar{E})\cdot\hat{n}\,dA$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

Maxwells suggestions

① $\oint \bar{E}\cdot\hat{n}\,dA = \frac{q}{4\pi\epsilon_0}$

② $\oint \bar{B}\cdot\hat{n}\,dA = \;?$

③ $\oint \frac{\partial B}{\partial t}\cdot\hat{n}\,dA = -\oint \bar{E}\cdot d\ell$

④ $\oint \bar{B}\cdot d\ell = \mu_0 I$ (?)

gauss law suggest that flux must end on electric charges - clearly $\oint \bar{E}\cdot\hat{n}\,dA = 0$ in regions with no net charge, this suggests

$$\oint \bar{B}\cdot\hat{n}\,dA = 0$$

that is equivalent to saying there no magnetic charges. This is a new Maxwell equation

Maxwell postulated that a time varying electric flux would create a magnetic field - he proposed replacing Ampere's law by

$$\oint B \cdot d\ell = \mu_0 \left(I + \epsilon_0 \frac{d}{dt} \oint \bar{E} \cdot \hat{n} \, dA \right)$$

which should be compared

$$\oint \bar{E} \cdot d\ell = -\frac{d}{dt} \oint \bar{B} \cdot \hat{n} \, dA$$

The the coefficient $\mu_0 \epsilon_0$ follows from dimensional analysis

$$\frac{B}{\mu_0} = \frac{Q}{d \cdot t} \qquad \epsilon_0 E \frac{A}{t} : \frac{Q}{d \cdot t}$$

In this case the induced magnetic field is in the opposite direction to the direction given by the right hand rule

$$\epsilon_0 \frac{d}{dt} \oint \bar{E} \cdot \hat{n} \, dA$$

is called the displacement current

the full set of Maxwells equations are

$$\oint \bar{E}\cdot\hat{n}\,dA = \frac{Q}{4\pi\epsilon_0}$$

$$\oint \bar{B}\cdot\hat{n}\,dA = 0$$

$$\oint \bar{B}\cdot d\bar{l} = \mu_0\left(I + \epsilon_0\int\frac{\partial E}{\partial t}\cdot\hat{n}\,dA\right)$$

$$\oint \bar{E}\cdot d\bar{l} = -\int\frac{\partial B}{\partial t}\cdot\hat{n}\,dA$$

The first and third equations involve current or charge sources; the second and fourth equations are called source free equations

Recall with Faradays law we showed

$$\oint \bar{E}\cdot d\bar{l} = \int(\bar{\nabla}\times\bar{E})\cdot\hat{n}\,dA$$

by looking at $\oint E\cdot dl$ around a small square. using the same method

$$\oint \bar{B}\cdot d\bar{l} = \int(\bar{\nabla}\times\bar{B})\cdot\hat{n}\,dA$$

These relations lead to

$$\oint \bar{E}\cdot d\ell = \int (\bar{\nabla}\times\bar{E})\cdot\hat{n}\, dA = \int -\left(\frac{\partial B}{\partial t}\cdot\hat{n}\right) dA$$

$$\oint \bar{B}\cdot d\ell = \mu_0 \int J\cdot n\, dA + \mu_0\epsilon_0 \int \frac{\partial E}{\partial t}\cdot\hat{n}\, dA = \int (\nabla\times B)\cdot\hat{n}\, dA$$

since the area enclosed by the loop is arbitrary

$$\boxed{\begin{aligned}\bar{\nabla}\times\bar{E} &= -\frac{\partial\bar{B}}{\partial t}\\ \bar{\nabla}\times\bar{B} &= \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t} + \mu_0\bar{J}\end{aligned}}$$

It is useful to derive the differential form of these equations

① consider a small cube centered $\bar{r}_0 = (x y z)$ with sides of length Δx Δy Δz

$$\oint \bar{E}\cdot\hat{n}\, dA$$

In this integral $\bar{E}(r) = \bar{E}(r_0) +$ corrections $\propto \Delta x$ Δy or Δz

$$\oint \bar{E}\cdot\hat{n}\, dA = \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} \int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} \left(E_x(x+\Delta x/2, y\, z) - E_x(x-\tfrac{\Delta x}{2}, y\, z)\right) +$$

$$\int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} \int_{x-\Delta x/2}^{x+\Delta x/2} \left(E_y(x, y+\tfrac{\Delta y}{2}, z) - E_y(x, y-\tfrac{\Delta y}{2}, z)\right)$$

$$\int_{x-\Delta x/2}^{x+\Delta x/2} \int_{y-\Delta y/2}^{y+\Delta y/2} \left(E_z(x y\, z+\tfrac{\Delta z}{2}) - E_z(x y\, z-\tfrac{\Delta z}{2})\right)$$

$$= \Delta y \Delta z \frac{\partial E_x}{\partial x} \Delta x + \Delta z \Delta x \frac{\partial E_y}{\partial y} \Delta y + \Delta x \Delta y \frac{\partial E_z}{\partial z} \Delta z$$

$$= \int d^3r \, \bar{\nabla} \cdot \bar{E} = \frac{Q}{\epsilon_0} = \int \frac{\rho(r)}{\epsilon_0} d^3 r$$

this means coulombs law become

$$\boxed{\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0}}$$

and by the same calculation

$$\oint \bar{B} \cdot \hat{n} \, dA \Rightarrow \bar{\nabla} \cdot \bar{B} = 0$$

$$\boxed{\bar{\nabla} \cdot \bar{B} = 0 \, .}$$

Recall the equations for the line integral around a small square give

$$\oint B \cdot d\ell = \int_{x-\Delta/2}^{x+\Delta/2} \left(B_x(x', y - \tfrac{\Delta}{2} \right) dx' + \int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} B_y(x + \tfrac{\Delta}{2}, y') \, dy$$

$$\int_{x+\Delta x/2}^{x-\Delta x/2} B_x(x', y + \tfrac{\Delta}{2}) dx' + \int_{y+\frac{\Delta y}{2}}^{y-\frac{\Delta y}{2}} B_y(x - \tfrac{\Delta x}{2}, y' z) \, dy$$

$$- B_x(x, y + \Delta/2) \Delta x + B_x(x, y - \Delta/2) \Delta x$$

$$+ B_y(x + \tfrac{\Delta}{2} y) \Delta y - B_y(x - \tfrac{\Delta}{2} y) \Delta y + O(\Delta^2)$$

$$\left(- \frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x} \right) \Delta x \Delta y + \cdots$$

$$(\nabla \times B)_z$$

This leads to the differential form of maxwells equations

$$\boxed{\begin{aligned} \bar{\nabla}\cdot\bar{E} &= \rho/\epsilon_0 \\ \bar{\nabla}\cdot\bar{B} &= 0 \\ \bar{\nabla}\times\bar{E} &= -\frac{\partial B}{\partial t} \\ \bar{\nabla}\times\bar{B} &= \mu_0\epsilon_0 \frac{\partial E}{\partial t} + \mu_0 J \end{aligned}}$$

Recall

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

This holds if we replace A, B C by $\bar{\nabla}$ provided everything the differential is placed on the right

$$\underset{A}{\bar{\nabla}}\times(\underset{B}{\bar{\nabla}}\times\underset{C}{\bar{E}}) = \underset{B}{\bar{\nabla}}(\underset{A}{\bar{\nabla}}\cdot\underset{C}{\bar{E}}) - \underset{(A\cdot B)}{(\bar{\nabla}\cdot\bar{\nabla})}\underset{\bar{C}}{\bar{E}} = -\bar{\nabla}\cdot\bar{\nabla} E$$

$$\nabla\times(\nabla\times\bar{B}) = \nabla(\bar{\nabla}\cdot\bar{B}) - (\nabla\cdot\nabla)\bar{B} = -\bar{\nabla}\cdot\bar{\nabla} B$$

since $\bar{\nabla}\cdot\bar{E}$ and $\bar{\nabla}\cdot\bar{B} = 0$ in regions where there are no charges and currents

$$\nabla\times(\bar{\nabla}\times\bar{E}) = -\bar{\nabla}\cdot\bar{\nabla}E = -\frac{\partial}{\partial t}(\nabla\times B) = -\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\nabla\times(\bar{\nabla}\times\bar{B}) = -\bar{\nabla}\cdot\bar{\nabla} E \quad \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\times E = -\mu_0\epsilon_0\frac{\partial^2 B}{\partial t^2}$$

This shows that all components of both field satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{E} = 0 \qquad (1)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{B} = 0 \qquad (2)$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

has units $\frac{s^2}{m^2}$ $\quad c^2 = \frac{1}{\mu_0 \epsilon_0} = (\text{speed of light})^2$

$$\frac{1}{\mu_0 \epsilon_0} = \frac{4\pi}{\mu_0 4\pi\epsilon_0} = \frac{4\pi k}{\mu_0} = 10^7 \times 8.99 \times 10^9 = 8.99 \times 10^{16}\ \frac{m^2}{s^2}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8\ m/s$$

equations (1) and (2) are called wave equations - the solutions must still satisfy maxwell's equations.