Lecture 32

Maxwell's equations

① $\bar{E}(\bar{r}) = k \sum q_i \frac{\bar{r}-\bar{r}_i}{|\bar{r}-\bar{r}_i|^3}$ Coulomb's Law

$\oint \bar{E}(\bar{r}) \cdot \hat{n}\, dA = Q/\epsilon_0$ Gauss' Law

② $dB(\bar{r}) = \frac{\mu_0 I}{4\pi} \frac{d\bar{l}' \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$ Biot Savart Law

$\oint \bar{B} \cdot d\bar{l} = \mu_0 I$ Amperes law

$= \int \mu_0 \bar{J}(\bar{r}) \cdot \hat{n}\, dA$

③ $\frac{d}{dt} \int \bar{B}(\bar{r}) \cdot n\, dA = - \oint \bar{E}(\bar{r}) \cdot d\bar{l}$ Faradays Law

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There are 2 missing elements

$\oint \bar{B}(r) \cdot \hat{n}\, dA = ?$

* from the Biot Savart law the magnetic field falls off like $1/r^2$ like the electric field

* electric field lines end on charges; there are no magnetic field lines.

$$\boxed{\oint \bar{B}\cdot\hat{n}\,dA = 0}$$

This is called Gauss' law for magnetic fields, what it means is that there is no net flux in a closed surface.

This is a new law - it would have to be modified if there were magnetic charges

comparison

$$\oint \bar{B}(r)\cdot d\bar{l} = \mu_0 I$$

$$\oint \bar{E}(r)\cdot d\bar{l} = -\int \frac{\partial \bar{B}}{\partial t}(\bar{r})\cdot\hat{n}\,dA$$

Maxwell proposed the following modification to ampere's law

$$\boxed{\oint B(\bar{r})\cdot d\bar{l} = \mu_0\left(I + \epsilon_0 \int \frac{\partial E}{\partial t}(r)\cdot\hat{n}\,dA\right)}$$

$\epsilon_0 \int \frac{\partial E}{\partial t}(r)\cdot\hat{n}\,dA$ is called Maxwell's displacement current.

Maxwells motivation involved studying fields in dielectric media

The coefficient $\mu_0 \epsilon_0$ is needed for dimensional reasons.

$$F = qE = qv \times B$$

dimensions $\left(\frac{E}{B}\right) = V = \frac{m}{s}$

Biot Savart

$$\mu_0 \sim \frac{B}{C} \cdot s\, d \qquad \epsilon_0 \sim \frac{1}{E}\, C\, \frac{1}{d^2}$$

$$\mu_0 \epsilon_0 \sim \frac{B}{E} \frac{s \cdot d}{C} \cdot \frac{C}{d^2} = \frac{s}{d} \cdot \frac{B}{E} = \frac{s}{d} \cdot \frac{s}{d} = \frac{1}{(\text{speed})^2}$$

$$\frac{1}{\mu_0 \epsilon_0} = \frac{4\pi}{\mu_0} \cdot \frac{1}{4\pi \epsilon} = \frac{4\pi}{\mu} \cdot k = 10^7 \times 8.99 \times 10^9 = 8.99 \times 10^{16}$$

$$= c^2 \ (\text{speed of light})^2$$

There is a differential form for each of these laws. We will derive them from the integral relations

To determine the differential form of Maxwells equations we use the integral form over small volumes

consider

$$f(x) = f(x_0) + \frac{df}{dx}(x_0)(x-x_0) + \frac{1}{2}\frac{d^2f}{dx^2}(x_0)(x-x_0)^2 + \cdots$$

$$\int_{x_0}^{x_0+\Delta} f(x)\,dx =$$

$$f(x_0)\Delta + \frac{df}{dx}(x_0)\frac{1}{2}\Delta^2 + \frac{1}{2}\frac{d^2f}{dx^2}(x_0)\frac{1}{3}\Delta^3$$

$$f(x_0)\Delta + \frac{1}{2}\frac{df}{dx}(x_0)\Delta^2 + \frac{1}{6}\frac{d^2y}{dx^2}(x_0)\Delta^3 +$$

smaller terms

consider a surface integral around a cube of side Δ

$$\int \bar{E}\cdot\hat{n}\,dA = \Sigma \text{ 6 terms}$$

$$\int_{y_0}^{y_0+\Delta} dy \int_{x_0}^{x_0+\Delta} dx \left(E_z(x, y\ z_0+\Delta) - E_z(x\ y\ z_0)\right) +$$

$$\int_{y_0}^{y_0+\Delta} dy \int_{z_0}^{z_0+\Delta} dz \left(E_x(x_0+\Delta\ y\ z) - E_x(x_0\ y\ z)\right) +$$

$$\int_{x_0}^{x_0+\Delta} dy \int_{z_0}^{z_0+\Delta} dz \left(E_y(y_0+\Delta, y\ z) - E_y(y_0\ y\ z)\right) =$$

$$\int_{Y_0}^{Y_0+\Delta} dy \int_{X_0}^{X_0+\Delta} dx \int_{Z_0}^{Z_0+\Delta} \frac{\partial E_z}{\partial z}(yzz)\,dz$$

$$\int_{Y_0}^{Y_0+\Delta} dy \int_{Z_0}^{Z_0+\Delta} dz \int_{X_0}^{X_0+\Delta} dx \quad \frac{\partial E_x}{\partial x}(xyz)\,dx$$

$$\int_{X_0}^{X_0+\Delta} dx \int_{Z_0}^{Z_0+\Delta} dz \int_{Y_0}^{Y_0+\Delta} dy \quad \frac{\partial E_y}{\partial y}(xyz) =$$

$$\underbrace{\int dx\,dy\,dz}_{\text{volume of cube}} \quad \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)(xyz) =$$

$$\Delta^3 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Big|(x_0 y_0 z_0) + (\quad)\,\Delta^4 + \cdots$$

For Gauss law

$$\int \bar{E}\cdot\hat{n}\,dA = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho(xyz)\,dx\,dy\,dz =$$

$$\frac{1}{\epsilon_0}\Delta^3 \rho(x_0 y_0 z_0) + \frac{1}{\epsilon_0}(\quad)\,\Delta^4$$

divide both sides by Δ^3 and let $\Delta^3 \to 0$

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)(x_0 y_0 z_0) - \frac{1}{\epsilon_0}\rho(x_0 y_0 z_0) + (\quad)\underbrace{\Delta}_{\to 0}$$

as $\Delta \to 0$ we get

$$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{1}{\epsilon_0}\rho}$$

If we replace $\bar{E}(xyz)$ by $\bar{B}(xyz)$ and note $\oint \bar{B}\cdot\hat{n}\, dA = 0$ we get

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0}$$

which is the differential form of Gauss law for magnetic fields

Next consider

$$\oint \bar{B}\cdot d\bar{l} = \mu_0\left(I + \epsilon_0 \int \frac{\partial E}{\partial t}\cdot\hat{n}\, dA\right)$$

in this case we integrate around a small square

$$\oint \bar{B}\cdot d\bar{l} = \int_{x_0}^{x_0+\Delta} B_x(x\, y_0)\, dx +$$

$$\int_{y_0}^{y_0+\Delta} B_y(x_0+\Delta, y) + \int_{x_0+\Delta}^{x_0} B_x(x, y_0+\Delta)\, dx +$$

$$\int_{y_0+\Delta}^{y} B_y(x_0, y) =$$

$$-\int_{x_0}^{x_0+\Delta} (B_x(x\, y_0+\Delta) - B_x(x\, y_0))\, dx$$

$$+\int_{y_0}^{y_0+\Delta} (B_y(x_0+\Delta, y) - B_y(x_0\, y))\, dy =$$

$$= \int_{x_0}^{x_0+\Delta} dx \int_{y_0}^{y_0+\Delta} \left(-\frac{\partial B_x}{\partial y}(xy) + \frac{\partial B_y}{\partial x}(xy) \right)$$

$$= \int_{x_0}^{x_0+\Delta} dx \int_{y_0}^{y_0+\Delta} \underbrace{\left(\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x \right)}_{(\bar{\nabla} \times \bar{B})_z}$$

this has the law

$$\int (\bar{\nabla} \times B) \cdot \hat{n} \, dA = \oint \bar{B} \cdot d\bar{l}$$

this is

$$\mu_0 \left(I + \epsilon_0 \int \frac{\partial E}{\partial t} \cdot \hat{n} \, dA \right)$$

$$\int \left(\mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \right) \cdot \hat{n} \, dA$$

putting everything togeth

$$0 = \int \left(-\bar{\nabla} \times \bar{B} + \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t} \right) \cdot \hat{n} \, dA = 0$$

over a small area if we divide by Δ^2
and take the limit $\Delta \rightarrow$ we get

$$\boxed{\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{\partial \bar{E}}{\partial t}}$$

If we do the same to Faraday's law

$$\oint \bar{E}\cdot d\bar{l} = \int (\bar{\nabla}\times\bar{E})\cdot\hat{n}\,dA = -\int \frac{\partial B}{\partial t}\cdot\hat{n}\,dA$$

which leads to

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Taken together we get –

using

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} =$$

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right) =$$

$$\bar{\nabla}\cdot\bar{E}$$

① $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$ (Differential form of Coulomb's or Gauss Law)

$\bar{\nabla}\cdot\bar{B} = 0$ (Differential form of Gauss Law for magnetism)

$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$ (Differential form of Faraday's Law)

$\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \mu_0\epsilon_0\frac{\partial \bar{E}}{\partial t}$ (Differential form of Ampere's Law with displacement current) we see that there are several equivalent forms of these 4 laws.

the equations $\bar{\nabla}\cdot\bar{B} = 0$ $\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$ are called the source free Maxwell equations. They would look more like the other 2 equation if there were magnetic charges and currents

Some manipulations

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

This is the bac-cab rule for triple cross products. It works for any vectors

if one of the vectors is $\bar{\nabla}$ we can use it but we need to keep the correct order

$$\vec\nabla \times (\vec A \times \vec B) = \vec\nabla(\overbrace{\vec A\ \vec B}) - \vec\nabla(\overbrace{\vec B\ \vec A})$$

$$= \sum \frac{\partial}{\partial x_i}(\vec A B_i) - \frac{\partial}{\partial x_i}(A_i \vec B)$$

$$\vec A \times (\vec\nabla \times \vec B) = \overbrace{\vec A \cdot \vec\nabla\ \vec B} - (\vec A \cdot \vec\nabla)\vec B$$

$$= \sum A_i \vec\nabla B_i - (\vec A \cdot \vec\nabla)\vec B$$

$$\vec A \times (\vec B \times \vec\nabla) = \vec B(\vec A \cdot \vec\nabla) - (A \cdot B)\vec\nabla$$

The important observation
we need to keep track of with
object the derivative act on -
and then use the chain rule

$$\vec\nabla \times (\vec\nabla \times \vec B) = \vec\nabla(\vec\nabla \cdot B) - (\vec\nabla \cdot \vec\nabla)\vec B$$

$$\vec\nabla \times (\vec B \times \vec\nabla) = \sum \vec\nabla_i(\vec B\ \nabla_i) - (\vec\nabla \cdot \vec B)\vec\nabla$$

etc

$$(\vec\nabla \cdot \vec B) f = f(\vec\nabla \cdot \vec B) + \vec B \cdot \vec\nabla f$$

i.e. we still have to use the chain rule

Lets apply this to maxwells equations in a region of space where $\rho(\bar{r}t) = \bar{J}(\bar{r}t) = 0$

$$\bar{\nabla}\cdot\bar{E} = 0 \qquad \bar{\nabla}\times\bar{B} = \mu_0\epsilon_0 \frac{\partial \bar{E}}{\partial t}$$
$$\bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = \nabla(\bar{\nabla}\cdot E) - (\bar{\nabla}\cdot\bar{\nabla})\bar{E}$$
$$\shortparallel$$
$$\bar{\nabla}\times\left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{B}) =$$
$$-\frac{\partial}{\partial t}\left(\mu_0\epsilon_0 \frac{\partial \bar{E}}{\partial t}\right) = -\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\bar{E}$$

since $\bar{\nabla}\cdot\bar{E} = 0$ this gives and

$\mu_0\epsilon_0 = \frac{1}{c^2}$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{E} = 0}$$

similarly

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{B}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{B}) - \bar{\nabla}\cdot\bar{\nabla}B =$$
$$\bar{\nabla}\times\left(\mu_0\epsilon_0 \frac{\partial \bar{E}}{\partial t}\right) = \mu_0\epsilon_0 \frac{\partial}{\partial t}(\bar{\nabla}\times\bar{E})$$
$$-\mu_0\epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial t}\right)$$

because $\bar{\nabla}\cdot\bar{B}=0$ this becomes

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{B}=0}$$

these equations are called wave equations. these are called wave equations

note while the electric and magnetic fields separately satisfy these equations, they are not independent - they are related by the 4 maxwells equations

consider $f(\bar{x}\cdot\hat{n}-ct)$

$$\bar{\nabla} f = f'\cdot\hat{n}$$

$$\bar{\nabla}\cdot\bar{\nabla} f = \bar{\nabla} f'(\bar{x}\cdot\hat{n}-ct)\cdot\hat{n} = f''\,\hat{n}\cdot\hat{n}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} f'(-c) = f''(c^2)$$

$$\left(\bar{\nabla}\cdot\bar{\nabla}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)f = f''\left(\hat{n}\cdot\hat{n}-\frac{c^2}{c^2}\right)=0$$

this is one of many solutions of the wave equation

$u = \hat{n}\cdot\bar{x} - ct$

for u constant

$$\hat{n}\cdot\frac{d\bar{x}}{dt} = c$$

which corresponds to a disturbance moving in the $\hat{n}$ direction at the velocity of light

consider a solution of the form

$$\bar{E} = \bar{E}_o \cos(\bar{x}\cdot\hat{n} - ct)$$

$$0 = \bar{\nabla}\cdot\bar{E} = \bar{E}_o\cdot\bar{\nabla}\cos(\bar{x}\cdot\hat{n} - ct)$$

$$= -\bar{E}_o\cdot\hat{n}\sin(\bar{x}\cdot\hat{n} - ct)$$

for this to hold $\bar{E}_o$ must be $\perp$ to the direction of motion

$$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t} =$$

$$\bar{\nabla}\times\bar{E}_o\cos(\bar{x}\cdot\hat{n} - ct) =$$

$$-\bar{E}_o\times\bar{\nabla}\cos(\bar{x}\cdot\hat{n} - ct) = \bar{E}_o\times\hat{n}\sin(\bar{x}\cdot\hat{n} - ct)$$

$$= +\frac{\partial}{\partial t}\frac{1}{c}\bar{E}_o\times\hat{n}\cos(\bar{x}\cdot\hat{n} - ct)$$

$$= -\frac{\partial B}{\partial t}$$

comparing gives

$$\bar{B} = \frac{1}{c}\,\hat{n} \times \bar{E}_0 \cos(\bar{x}\cdot\hat{n} - ct) = \bar{B}_0 \cos(\bar{x}\cdot\hat{n} - ct)$$

which shows that $\bar{B}_0$ is $\perp$ to both $\bar{E}_0$ and the direction of propagation $\hat{n}$ and $|E_0| = c|B_0|$