Lecture 36

Maxwells equations and electromagnetic waves

Last time we derived differential forms of Maxwells equations

$$\bar{E}(\bar{r}) = k \sum q_i \frac{\bar{r}-\bar{r}_i}{|r-r_i|^3}$$

$$\oint_A \bar{E}\cdot\hat{n}\,dA = \frac{q}{\epsilon_0}$$

$$\bar{\nabla}\cdot\bar{E}(\bar{r}) = \rho(r,t)/\epsilon_0$$

$$d\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi} \frac{d\bar{l}\times(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$\oint \bar{B}\cdot d\bar{l} = \mu_0 I + \mu_0\epsilon_0 \oint \frac{\partial\bar{E}}{\partial t}\cdot\hat{n}\,dA$$

$$\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \mu_0\epsilon_0\frac{\partial\bar{E}}{\partial t}$$

$$\oint \bar{E}\cdot dl = -\oint \frac{\partial B}{\partial t}\cdot\hat{n}\,dA$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\oint \bar{B}(r)\cdot\hat{n}\,dA = 0$$

$$\bar{\nabla}\cdot\bar{B}(r) = 0$$

each box represents equivalent forms of Maxwells equations

note

$$\bar{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\bar{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\begin{aligned}\bar{A} \times \bar{B} = \; & A_x \hat{x} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) \\ & + A_y \hat{y} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z}) \\ & + A_z \hat{z} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})\end{aligned}$$

$$\begin{aligned}= \; & (A_x B_y \hat{z} - A_x B_z \hat{y} \\ & - A_y B_x \hat{z} + A_y B_z \hat{x} \\ & A_z B_x \hat{y} - A_z B_y \hat{x}\end{aligned}$$

$$\begin{aligned}= \; & (A_y B_z - A_z B_y) \hat{x} + \\ & (A_z B_x - A_x B_z) \hat{y} + \\ & (A_x B_y - A_y B_x) \hat{z}\end{aligned}$$

next consider

$$\bar{A} \times (\bar{B} \times \bar{C}) =$$

$$\begin{aligned}& (A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)) \hat{x} + \\ & A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_x)) \hat{y} + \\ & A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)) \hat{z}\end{aligned}$$

$$\left(B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z)\right)\hat{x}$$
$$(+B_xA_xC_x - C_xA_xB_x)\hat{y} \quad +$$
$$(B_y(A_zC_z + A_xC_x) - C_y(A_zB_z + A_xB_x))\hat{y}$$
$$(+B_yA_yC_y - C_yA_yB_y)\hat{y}$$
$$(B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y))\hat{z}$$
$$(B_zA_zC_z - C_zA_zB_z)\hat{z} =$$

$$\bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

This identity is called the BCA - CAB rule

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$$

if we consider

$$\bar{\nabla}(fg) = g\bar{\nabla}f + f\bar{\nabla}g$$

chain rule

$$gf\bar{\nabla} = gf\bar{\nabla}$$

If we consider

$$\bar{\nabla}\times(\bar{B}\times\bar{C}) = \sum \nabla_i(\bar{B}C_i) - \sum\nabla_i(\bar{C}B_i)$$

we always have to place the differential operator so it acts only on everything to is right in the original expression

$$\boxed{\nabla\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \bar{\nabla}\cdot\bar{\nabla}\bar{A}}$$

here we see $\bar{A}$ is on the right of all of the derivates on both sides of the equation.

We use this in Maxwells equation

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\times\left(-\frac{\partial\bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{B})$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{E}) - \bar{\nabla}\cdot\bar{\nabla}\bar{E} \qquad\qquad -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right)$$

(here we assume we are in a region where $\bar{J}(\bar{r}t)=0$, $\rho(\bar{r}t)=0$

If $\rho(rt)=0$ then $(\bar{\nabla}\cdot\bar{E})=0$

what remains is

$$\left(-\bar{\nabla}\cdot\bar{\nabla} + \frac{\partial^2}{\partial t^2}\right)\bar{E}(\bar{r},t) = 0$$

similarly

$$\nabla\times(\bar{\nabla}\times\bar{B}) = \mu_0\epsilon_0\nabla\times\frac{\partial E}{\partial t} = \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\times E) =$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{B}) - (\bar{\nabla}\cdot\bar{\nabla})\bar{B} \qquad\qquad -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial t}\right)$$

since $\bar{\nabla}\cdot\bar{B}=0$

$$\left(-\bar{\nabla}\cdot\nabla + \mu_0\epsilon_0\frac{\partial^2}{\partial t}\right)\bar{B}(rt) = 0$$

This shows that both fields satisfy the wave equation

$$\left(\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) \begin{Bmatrix} \bar{E}(\bar{x},t) \\ \bar{B}(\bar{x}t) \end{Bmatrix} = 0$$

It looks like there are 6 independent solutions, but that is not true because the fields must also satisfy maxwells equations

since $\mu_0 \epsilon_0 = \frac{1}{c^2}$ this can be written

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) \begin{Bmatrix} \bar{E}(xt) \\ \bar{B}(xt) \end{Bmatrix} = 0$$

This is a partial differential equation. To understand solutions let $f(s)$ be any differentiable function

Let $\hat{n}$ be any unit vector and consider

$$f(\bar{x}\cdot\hat{n} \pm ct)$$

$$\frac{\partial f}{\partial t} = \pm c f'(\bar{x}\cdot\hat{n} \pm ct)$$

$$\frac{\partial^2 f}{\partial t^2} = (\pm c)^2 f''(\bar{x}\cdot\hat{n} \pm ct) = c^2 f''(\bar{x}\cdot\hat{n} \pm ct)$$

$$\left(f'' = \frac{d^2 f}{d s^2}\right)$$

$$\frac{\partial f}{\partial x} = n_x f'(\hat{n}\cdot\bar{x} \pm ct)$$

$$\frac{\partial^2 f}{\partial x^2} = n_x^2 f''(\hat{n}\cdot\bar{x} \pm ct)$$

etc

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) f(\hat{n}\cdot\bar{x} \pm ct) =$$

$$\underbrace{\left(\frac{1}{c^2}c^2 - n_x^2 - n_y^2 - n_z^2\right)}_{0} f(\hat{n}\cdot\bar{x} \pm ct) = 0$$

so we see there are many solutions

for $s = \hat{n}\cdot x \pm ct = $ constant

$$0 = \hat{n}\cdot\frac{dx}{dt} = \mp c$$

corresponds to a disturbance moving in the $\pm\hat{n}$ direction with speed c

Consider

$$\bar{E}(\bar{r},t) = \bar{E}_0 \cos(\hat{n}\cdot\bar{x} - ct)$$

This is clearly a solution of the wave equation since each component is for the form $f(\hat{n}\cdot\bar{x} - ct)$

Lets consider what Maxwells equations say

$$\bar{\nabla}\cdot\bar{E} = 0 =$$

$$\bar{\nabla}\cdot\bar{E}_0 \cos(\hat{n}\cdot\bar{x} - ct) =$$

$$\bar{E}_0(\hat{x}n_x + \hat{y}n_y + \hat{z}\hat{n}_z)(-\sin(\hat{n}\cdot\bar{x} - ct))$$

This means

$$\boxed{\bar{E}_0\cdot\hat{n} = 0}$$

so the direction of the field is $\perp$ to the direction of motion

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\shortparallel$$

$$-\bar{E}_0\times\bar{\nabla}\cos(\hat{n}\cdot\bar{x} - ct) =$$

$$-E_0\times\hat{n}(-\sin(\bar{n}\cdot\bar{x} - ct)) =$$

$$c\,\bar{B}_0(-\sin(\bar{n}\cdot x - ct))$$

comparing these expressions

$$\boxed{\bar{B}_0 = \frac{1}{c}\,\hat{n}\times\bar{E}_0}$$

This means

$$\boxed{\begin{array}{l}\bar{B}_0\cdot\hat{n} = 0\\ \bar{B}_0\cdot\bar{E}_0 = 0\end{array}}$$

in this example the electric and magnetic fields are $\perp$ to the direction of propagation and perpendicular to each other.

we can replace

$$f(\hat{n}\cdot\bar{x} \pm ct)$$

by

$$f(k\hat{n}\cdot\bar{x} \pm kct)$$

when we differentiate all of the previous derivations of the wave equation have an additional factor of k^2, but we have a solution of the wave equation

$$\bar{E} = E_0 \cos(k\hat{n}\cdot\bar{x} - kct)$$

is a solution of Maxwell's equation.

where

$\bar{k} = k\hat{n}$ = wave vector

k = wave number

$w = ct$ = angular frequency

Maxwell's equations still require

$\bar{E} \perp \bar{B} \quad \bar{E} \perp \bar{k} \quad \bar{B} \perp \bar{k}$

(the wave vector points in the direction of propagation)

$$k\lambda = 2\pi \implies \lambda = \frac{2\pi}{k} = \text{wavelength}$$

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega} = \text{period}$$

$$f = \frac{1}{T} \qquad f = \frac{\omega}{2\pi} = \text{frequency}$$

the solutions of the wave equation satisfy the superposition principle (because the equation is linear)

$$\bar{E}(xt) = \bar{E}_1 \cos(k_1 \hat{n}_1 \cdot \bar{x} - k_1 ct) + \bar{E}_2 \cos(k_2 \hat{n}_2 \cdot \bar{x} - k_2 ct)$$

is a solution if $\bar{E}_1(xt)$ and $\bar{E}_2(xt)$ are solutions - this follows by putting them in the wave equation

Poynting Vector

$$\boxed{\bar{S} = \frac{1}{\mu_0}\bar{E}\times\bar{B}}$$

$$B = \frac{1}{c}\,\hat{n}\times\bar{E}$$

$$\bar{E}\times\bar{B} = \frac{1}{c}\,E\times(\hat{n}\times E) = \frac{1}{c}\left(\hat{n}\,E^2 - \bar{E}\cdot(\underbrace{\hat{n}\cdot\bar{E}}_{0})\right)$$

$$= \frac{1}{c}\,\hat{n}\,E^2$$

$$\bar{S} = \frac{1}{\mu_0}\cdot\frac{1}{c}\,\hat{n}\,E^2 = \frac{1}{\mu\epsilon_0}\frac{1}{c}\,\epsilon_0\hat{n}E^2 = c\hat{n}\,\epsilon_0 E^2$$

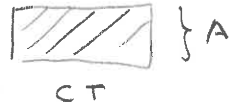
This is a vector that points in the direction of the electromagnetic wave

The energy density in a magnetic field is

$$\frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\frac{E^2}{c^2} = \frac{1}{2}\frac{\mu_0\epsilon_0}{\mu_0}E^2$$

$\bar{S}$ can be expressed as

$$\bar{S} = \underbrace{c\hat{n}}_{\text{speed of wave}}\left(\underbrace{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2}_{\text{energy density}}\right)$$



$\boxed{\bar{S}\cdot A\,cT = \text{energy deposited in area } A \text{ in time } T}$

$\int \bar{S}\cdot\hat{n}\,dA$ = energy deposited in Area A / time more generally

$$\bar{E}(\bar{x},t) = \sum_i \bar{E}_i \, e^{i(k_i \hat{n}_i \cdot \bar{x} - kct)}$$

is a solution

$$B(\bar{x}t) = \frac{1}{c} \sum \hat{n}_i \times \bar{E}_i \, e^{i(k_i \hat{n} \cdot \bar{x} - kct)}$$

Just like with electric charges -
using $\omega = kc$ $\quad k = \frac{\omega}{c}$ $\quad \bar{k} = \hat{n} k$

$$\bar{E}(xt) = \int \mathcal{E}(\bar{k}) \, e^{i(k \cdot \bar{x} - |\bar{k}|ct)} \, d^3k$$

$$\bar{B}(xt) = \frac{1}{c} \int \hat{k} \times \bar{\mathcal{E}}(k) \, e^{i(\bar{k} \cdot \bar{x} - |k|ct)} \, d^3k$$

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It turns out that any electromagnetic wave can be put in the form *