Lecture 35

Electromagnetic Waves

Last time

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{E}(\bar{x},t) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{B}(\bar{x}\,t) = 0$$

solutions

Let $f(s)$ be any function that has 2 derivatives

$$f(\hat{n}\cdot\bar{r} \pm ct)$$

when $s = \text{constant} = \hat{n}\cdot\bar{r} \pm ct$

$$0 = \frac{ds}{dt} = \hat{n}\cdot\frac{d\bar{r}}{dt} = \mp c$$

corresponds to a disturbance moving in direction $\pm\hat{n}$ with speed c

consider $f(s) = \cos(ks)$

$$\bar{E}(\bar{r}t) = \bar{E}_0 \cos(k\hat{n}\cdot\bar{r} - kct)$$

we define

$\bar{k} = k\hat{n}$ = wave vector

$|\bar{k}|$ = wave number

$\omega = kc$ = angular frequency

we use Maxwells equations to ~~derive~~ determine properties of constants

① $\bar{\nabla}\cdot\bar{E} = 0$

$\bar{E}_0\cdot\bar{k}\,(-\sin(\bar{k}\cdot\bar{r}-\omega t)) = 0$

This requires $\bar{E}_0\cdot\bar{k} = 0$

② $\frac{\partial\bar{B}}{\partial t} = -\bar{\nabla}\times E \Rightarrow$

$\frac{\partial\bar{B}}{\partial t} = -\bar{k}\times\bar{E}_0(-\sin(\bar{k}\cdot\bar{r}-\omega t))$

$= \frac{\partial}{\partial t}\,\bar{k}\times\bar{E}_0\frac{1}{\omega}\cos(\bar{k}\cdot\bar{x}-\omega t)$

This gives

$\bar{B}(t) = (\frac{1}{\omega}\bar{k}\times\bar{E}_0\cos(\bar{k}\cdot\bar{x}-\omega t))$

$= (\frac{1}{c}\hat{n}\times\bar{E}_0\cos(\bar{k}\cdot\bar{x}-\omega t)$

We note that $\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \bar{B}_0\cdot\hat{n} = 0$ which is automatic in this case

There is nothing special about using $\cos(\bar{k}\cdot\bar{x}-\omega t)$

We could also use

$$\bar{E} = \bar{E}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

$$\bar{B} = \frac{1}{c}\hat{n}\times\bar{E}_0 e^{i(\bar{k}\cdot\bar{x}-\omega t)}$$

$$\bar{E}_0\cdot\hat{n} = 0$$

The wave equation is linear

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\bar{E}_1(\bar{r}t)+\bar{E}_2(\bar{r}t)\right) =$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{E}_1(r,t) +$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{E}_2(rt) = 0$$

This means if $E_1(\bar{r}t)$ and $E_2(rt)$ are solutions so is $E_1(rt)+E_2(rt)$.

more generally

$$\boxed{\begin{aligned}\bar{E}(\bar{r}t) &= \int d^3k\ \bar{E}_0(\bar{k})e^{i(\bar{k}\cdot\bar{r}-c|k|t)}\\ \bar{B}(\bar{r}t) &= \int d^3k\ \bar{B}_0(\bar{k})e^{i(\bar{k}\cdot\bar{r}-c|k|t)}\\ \bar{B}_0(k) &= \frac{1}{c}\hat{k}\times\bar{E}_0(\bar{k})\end{aligned}}$$

It turns out any solution of Maxwell's equations can be expressed in this form

Energy Transport:

recall energy density

$$\text{energy} = \frac{\epsilon_0}{2} \bar{E}^2(\bar{r}t) + \frac{1}{2\mu_0} \bar{B}^2(rt)$$

recall the charge current is

$$\bar{J}(\bar{r}t) = q n(rt) \bar{v}(rt)$$

since E and B have the same factor of $\cos(\bar{k}\cdot\bar{x} - \omega t)$

energy current

$$\bar{J}(rt) = c\left(\frac{1}{2} \epsilon_0 \bar{E}^2 + \frac{1}{2\mu_0} \bar{B}^2 \right)$$

Claim

$$\bar{S} = \frac{1}{\mu_0} \bar{E} \times \bar{B} \quad \text{(Poynting vector)}$$

$$\bar{S} = \frac{1}{\mu_0} \bar{E}_0 \times \left(\frac{1}{c} \hat{n} \times \bar{E}_0\right) \cos^2(\bar{k}\cdot\bar{r} - \omega t)$$

$$= \frac{1}{c\mu_0} \left(\hat{n}\, \bar{E}_0 \cdot \bar{E}_0 - \bar{E}_0 \underbrace{(\hat{n}\cdot\bar{E}_0)}_{0}\right) \cos^2(\bar{k}\cdot\bar{r} - \omega t)$$

$$= \frac{\epsilon_0}{c\mu_0\epsilon_0} \hat{n}\, \bar{E}_0 \cdot \bar{E}_0 \cos^2(\bar{k}\cdot\bar{r} - \omega t)$$

$$= c\epsilon_0 \bar{E}^2$$

$$B_0^2 = \frac{1}{c^2} (\hat{n} \times \bar{E}) \cdot (\hat{n} \times \bar{E}) = \frac{E_0^2}{c^2}$$

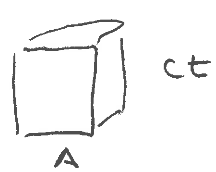
$$\bar{S} = \hat{n} c\, \epsilon_0 \left(\frac{1}{2} E^2 + \frac{1}{2} c^2 B^2\right)$$

$$= \hat{n} c \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2} \frac{\epsilon_0}{\epsilon_0 \mu_0} B^2\right)$$

$$= \hat{n} c \left(\frac{\epsilon_0}{2} \bar{E}^2 + \frac{1}{2\mu_0} \bar{B}^2\right)$$

This shows that the Poynting vector is exactly the energy current of an electromagnetic wave.

Power Absorbed



$$P = \int \bar{S} \cdot \hat{n}\, dA$$

$$= c(\text{energy density}) A$$

$$= \frac{d}{dt} \underbrace{(ct)(A)}_{\text{Volume}} \text{energy density})$$

For periodic waves

$$\bar{E}(rt) = \bar{E}_0 \cos(\bar{k}\cdot\bar{r} - \omega t)$$

$$\bar{E}(\bar{r}t)^2 = \bar{E}_0^2 \cos^2(\bar{k}\cdot\bar{r} - \omega t)$$

If we average this over 1 period

$$\langle \bar{E}^2 \rangle = \bar{E}_0^2 \frac{1}{T}\int_0^T \cos^2(\bar{k}\cdot\bar{r} - \omega t)\,dt$$

$$T = \frac{2\pi}{\omega} \qquad u = \omega t - \bar{k}\cdot\bar{r} \qquad du = \omega dt$$

$$\langle \bar{E}^2 \rangle = \bar{E}_0^2 \frac{\omega}{2\pi}\int_{-k\cdot r}^{-k\cdot r + 2\pi} \cos^2 u \cdot \frac{du}{\omega}$$

$$= \bar{E}_0^2 \frac{\omega}{\omega}\cdot\frac{2\pi}{2\pi}\cdot\frac{1}{2} = \frac{1}{2}\bar{E}_0^2$$

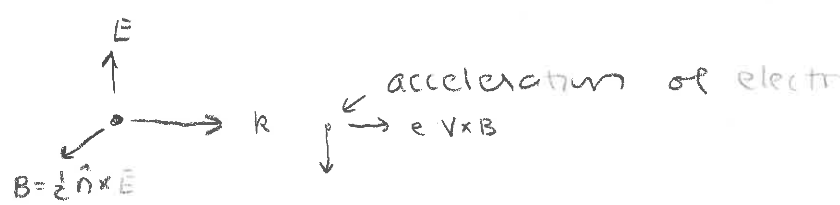
$$\bar{E}_{rms} = \frac{1}{\sqrt{2}}\hat{E}_0$$

$$\langle \bar{E}^2 \rangle = \bar{E}_{Rms}^2$$

The intensity of an electromagnetic wave is

$$I \equiv \langle S \rangle = \frac{1}{c\mu_0}\langle \bar{E} \rangle^2 = \frac{1}{c\mu_0} E_{rms}^2$$

Momentum



This shows that the electromagnetic exerts a force on the electron is the direction of propagation

Radiation pressure

Consider an electromagnetic wave incident on a surface of area A

Assume that the wave is absorbed

$$\frac{\Delta u}{\Delta t} = IA$$

average energy absorbed / time -

$\Delta u = F \Delta x = F c \Delta t$ change in energy in Δx

$$\frac{F c \Delta t}{\Delta t} = IA$$

$\frac{F}{A} = \langle \text{pressure} \rangle = \frac{IA}{c}$ absorption

If the radiation is reflected, then the momentum of the wave is reversed and by momentum conservation

$$\frac{F}{A} = 2\frac{IA}{c} = \langle \text{Pressure} \rangle$$

where this represents the average pressure electromagnetic energy

energy of 100 particles moving with speed v = energy of 1 particle moving with speed 10 v

$$(100)\frac{1}{2}mv^2 = \frac{1}{2}m(10v)^2 = (100)\frac{1}{2}mv^2$$

a similar phenomena occurs in electromagnetic waves - in quantum mechanics the waves are quantized. The elementary quanta are called photons. Each one has energy

$$\boxed{E = hf = \frac{h}{T} = h\frac{\omega}{2\pi} = \hbar\omega}$$

high frequency photons have more energy / photon than low frequency photons.

This is why x rays do more damage than radio waves.

polarization:

A wave of the form

$$\bar{E} = \bar{E}_0 \cos(\bar{k}\cdot\bar{x} - \omega t)$$

$$\bar{B} = \frac{1}{c}\hat{k}\times\bar{E}_0 \cos(\bar{k}\cdot\bar{x} - \omega t)$$

where $\bar{E}_0$ is a constant vector is called a linearly polarized wave

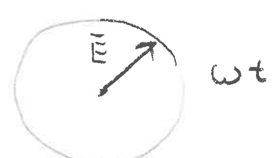
consider

$$\bar{E} = E(\hat{x}\cos(\bar{k}\cdot\bar{x} - \omega t) - \hat{y}\sin(\bar{k}\cdot\bar{x} - \omega t))$$

$$\bar{B} = \frac{1}{c} k\times E(\hat{x}\cos(k\cdot x - \omega t) - \hat{y}\sin(k\cdot x - \omega t))$$

For fixed $x=0$

$$\bar{E} = E(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$$



in this case in a plane of fixed $\bar{x}$, the direction of the field rotates

this is called a circularly polarized wave

In general a wave will be a superposition of waves with random polarizations. These are called unpolarized waves.
polarized sunglasses are transparent to waves that are linearly polarized in 1 direction

Magnetic properties of matter

recall in dielectric media

$$\int \bar{E}\cdot\hat{n}\, dA = q/\epsilon_0 \rightarrow q_s/\epsilon$$

w $q = q_{free} + q_{polarizat}$

$\epsilon = k\epsilon_0$

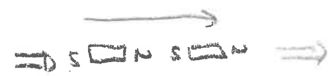
k = dielectric constant

For magnetic - in a medium with magnetic dipoles

$U = -\bar{\mu}\cdot\bar{B}$ dipoles lowest energy state is parallel to applied field

$$\bar{B}_{dip} = \frac{\mu_0}{4\pi}\left(\frac{3\bar{r}(\bar{\mu}\cdot\bar{r}) - \bar{\mu}r^2}{r^5}\right)$$

the dipoles generate an additional magnetic field — you can think of dipoles like little bar magnets



these tend to enhance the actual field

From the expression for the field due to a dipole

$$\frac{B_{dip}}{\mu_0} = \frac{\text{dipole moment}}{\text{volume}}$$

$\bar{M}$ = magnetization = dipole moment / volume

$\bar{B}$ = actual magnetic field

define $\bar{H}$ by

$$\bar{B} = \mu_0 \bar{H}$$

when there are no dipoles. When there are dipoles

$$\bar{B} = \mu_0 (\bar{H} + \bar{M})$$

here $\bar{H}$ is like the applied field in some care

$$M = \chi H$$

where χ is called the magnetic susceptibility

$$B = \mu_0 (H + \chi H) = \mu_0 (1+\chi) H$$
$$= \mu H$$

most materials have

$\chi > 0$ (paramagnetic materials)

some have

$\chi < 0$ (diamagnetic materials)

and some have a more complex H dependence (ferromagnetic)

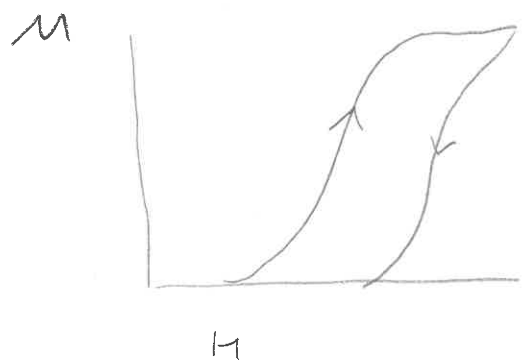
$$B = \mu_0 (\bar{H} + \bar{M})$$

x in paramagnetic materials have random dipoles that align with the fields

x id diamagnetic substances - Lenz's law - with complications from quantum mechanics reduces the field strength. This effect is generally small.

The last case the dipoles are remain aligned for a long time because they are in a lower energy state. Eventully thermal effects take over when the applied field is turned off.

For permanent magnets the magnetization is not a single valued function of H



This phenomena is called hysteresis

final remark - in media

$$\epsilon_0 \to \epsilon \geq \epsilon_0$$
$$\mu_0 \to \mu \geq \mu_0$$

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \qquad \frac{c^2}{n^2} = \frac{1}{\epsilon \mu}$$

The speed of light in a medium with electric and magnetic dipoles is $\frac{c}{n}$ where

$$n = \left(\frac{\epsilon \mu}{\epsilon_0 \mu_0}\right)^{1/2}$$

is called the index of refraction.