Lecture 35

Electromagnetic Waves

Last time

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E}(\vec{x}, t) = 0
\]

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B}(\vec{x}, t) = 0
\]

Solutions

Let \( f(s) \) be any function that has 2 derivatives

\[ f(\hat{n}. \vec{F} \pm ct) \]

where \( s = \text{constant} = \hat{n}. \vec{F} \pm ct \)

\[ 0 = \frac{ds}{dt} = \hat{n}. \frac{d\vec{F}}{dt} = \hat{n}. \vec{c} = rc \]

corresponds to a disturbance moving in direction \( \pm \hat{n} \) with speed \( c \)

Consider \( f(s) = \cos(ks) \)

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 \cos(k\hat{n}.\vec{F} - kct) \]

we define

\( \vec{k} = k\hat{n} \) = wave vector

\( k \) = wave number

\( \omega = kc \) = angular frequency
we use Maxwell's equations to derive determine properties of constants

\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \mathbf{E}_0 \cdot \mathbf{k} (-\sin (\mathbf{k} \cdot \mathbf{r} - \omega t)) = 0 \]

This requires \( \mathbf{E}_0 \cdot \mathbf{k} = 0 \)

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{r} \times \mathbf{E}_0 (-\sin (\mathbf{k} \cdot \mathbf{r} - \omega t)) \]
\[ = \frac{\partial}{\partial t} \mathbf{r} \times \mathbf{E}_0 \frac{1}{\omega} \cos (\mathbf{k} \cdot \mathbf{r} - \omega t) \]

This gives

\[ \mathbf{B}(t) = (\frac{1}{\omega} \mathbf{r} \times \mathbf{E}_0 \cos (\mathbf{k} \cdot \mathbf{r} - \omega t)) \]
\[ = (\frac{1}{c} \hat{\mathbf{r}} \times \mathbf{E}_0 \cos (\mathbf{k} \cdot \mathbf{r} - \omega t)) \]

We note that \( \nabla \cdot \mathbf{B} = 0 \) and \( \mathbf{B} \cdot \hat{\mathbf{n}} = 0 \)

which is automatic in this case.
There is nothing special about using $\cos (\vec{k} \cdot \vec{x} - \omega t)$

We could also use

$$\vec{E} = \vec{E}_0 \ e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\vec{B} = \frac{1}{c} \ \hat{n} \times \vec{E}_0 \ e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\vec{E}_0 \cdot \hat{n} = 0$$

The wave equation is linear

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) (\vec{E}_1 (r,t) + \vec{E}_2 (r,t)) =$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{E}_1 (r,t) +$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{E}_2 (r,t) = 0$$

This means if $\vec{E}_1 (r,t)$ and $\vec{E}_2 (r,t)$ are solutions, so is $\vec{E}_1 (r,t) + \vec{E}_2 (r,t)$.

More generally

$$\vec{E} (r,t) = \int d^3 k \ \vec{E}_0 (\vec{k}) \ e^{i(\vec{k} \cdot \vec{r} - c \lambda k t)}$$

$$\vec{B} (r,t) = \int d^3 k \ \vec{B}_0 (\vec{k}) \ e^{i(\vec{k} \cdot \vec{r} - c \lambda k t)}$$

$$\vec{B}_0 (\vec{k}) = \frac{1}{c} \ \hat{n} \times \vec{E}_0 (\vec{k})$$
It turns out any solution of Maxwell's equations can be expressed in this form:

**Energy Transport:**

Recall energy density:

\[ \text{energy} = \frac{E^2}{2} \hat{E}^2(r,t), \quad \frac{1}{2\mu_0} \hat{B}^2(r,t) \]

Recall the charge current is:

\[ \bar{J}(r,t) = qn(r) \bar{v}(r,t) \]

Since \( E \) and \( B \) have the same factor of \( \cos(kx - \omega t) \)

Energy current:

\[ \bar{J}(r,t) = c \left( \frac{1}{2} \varepsilon_0 \bar{E}^2 + \frac{1}{2\mu_0} \bar{B}^2 \right) \]

Claim:

\[ \bar{S} = \frac{1}{\mu_0} \bar{E} \times \bar{B} \] (Poynting vector)
\[ \bar{S} = \frac{1}{\mu_0} \bar{E}_0 \times \left( \frac{1}{c} \hat{n} \times \bar{E}_0 \right) \cos^3 \left( \hat{k} \cdot \bar{r} - \omega t \right) \]
\[ = \frac{1}{c \mu_0} \left( \hat{n} \bar{E}_0 \bar{E}_0 - \bar{E}_0 (\hat{n} \cdot \bar{E}_0) \right) \cos^3 \left( \hat{k} \cdot \bar{r} - \omega t \right) \]
\[ = \frac{\varepsilon_0}{c \mu_0 \varepsilon_0} \hat{n} \bar{E}_0 \bar{E}_0 \cos^3 \left( \hat{k} \cdot \bar{r} - \omega t \right) \]
\[ = \varepsilon_0 \bar{E}^2 \]
\[ B_o^2 = \frac{1}{c^2} (\hat{n} \times \bar{E}) (\hat{n} \times \bar{E}) = \frac{\varepsilon_0}{c^2} \]
\[ \bar{S} = \hat{n} c \varepsilon_0 \left( \frac{1}{2} \bar{E}^2 + \frac{1}{2} c^2 \bar{B}^2 \right) \]
\[ = \hat{n} c \left( \frac{\varepsilon_0}{2} \bar{E}^2 + \frac{1}{2} \frac{\varepsilon_0}{\mu_0} \bar{B}^2 \right) \]
\[ = \hat{n} c \left( \frac{\varepsilon_0}{2} \bar{E}^2 + \frac{1}{2 \mu_0} \bar{B}^2 \right) \]

This shows that the Poynting vector is exactly the energy current of an electromagnetic wave.

\[ \text{Power Absorbed} \]
\[ P = \int \bar{S} \cdot \hat{n} \, dA \]
\[ = C (\text{energy density}) A \]
\[ = \frac{d}{dt} \left( \frac{1}{V} (\text{energy density})(A) \right) \]
For periodic waves

\[ E(t) = E_0 \cos(\bar{k} \cdot \bar{r} - \omega t) \]
\[ E(t)^2 = E_0^2 \cos^2(\bar{k} \cdot \bar{r} - \omega t) \]

If we average this over one period

\[ \langle E^2 \rangle = E_0^2 \frac{1}{T} \int_{-T/2}^{T/2} \cos^2(\bar{k} \cdot \bar{r} - \omega t) \, dt \]

\[ T = \frac{2\pi}{\omega} \quad u = \omega t - \bar{k} \cdot \bar{r} \quad du = \omega \, dt \]

\[ \langle E^2 \rangle = E_0^2 \frac{\omega}{2\pi} \int_{-\bar{k} \cdot \bar{r}}^{\bar{k} \cdot \bar{r} + 2\pi} \cos^2 u \cdot \frac{du}{\omega} \]

\[ = E_0^2 \frac{\omega}{2\pi} \cdot \frac{2\pi}{2} = \frac{1}{2} E_0^2 \]

\[ E_{rms} = \sqrt{\langle E^2 \rangle} \]
\[ \langle E^2 \rangle = E_{rms}^2 \]

The intensity of an electromagnetic wave is

\[ I = \langle S \rangle = \frac{1}{c \mu_0} \langle E \rangle^2 = \frac{1}{c \mu_0} E_{rms}^2 \]

Momentum

\[ \vec{p} = \frac{E}{c} \]

acceleration of electron

\[ \vec{F} = e \vec{E} \times \vec{B} \]

This shows that the electromagnetic field exerts a force on the electron in the direction of propagation.
Radiation pressure

Consider an electromagnetic wave incident on a surface of area $A$.

Assume that the wave is absorbed.

$$\frac{\Delta U}{\Delta t} = IA$$

Average energy absorbed / time -

$\Delta U = F \Delta x = F c \Delta t$ change in energy in $\Delta x$

$$\frac{F c \Delta t}{\Delta t} = IA$$

$$\frac{F}{A} = \left< \text{pressure} \right> = \frac{IA}{c} \text{ absorption}$$

If the radiation is reflected, then the momentum of the wave is reversed and by momentum conservation

$$\frac{F}{A} = 2 \frac{IA}{c} = \left< \text{pressure} \right>$$

where this represents the average pressure.
electromagnetic energy

energy of 100 particles moving with speed \( V = \) energy of 1 particle moving with speed \( 10V \)

\[
(100)^{\frac{1}{2}} m v^2 = \frac{1}{2} m (10v)^2 = (100)^{\frac{1}{2}} m v^2
\]

a similar phenomena occurs in electromagnetic waves - in quantum mechanics the waves are quantized. The elementary quanta are called photons. Each one has energy

\[
E = hf = \frac{h}{\tau} = h \frac{\omega}{2\pi} = \hbar \omega
\]

high frequency photons have more energy / photon than low frequency photons.

This is why \( X \) rays do more damage than radio waves.
polarized

A wave of the form

\[ \mathbf{E} = \mathbf{E}_0 \cos (k \cdot \mathbf{x} - \omega t) \]
\[ \mathbf{B} = \frac{1}{c} \mathbf{k} \times \mathbf{E}_0 \cos (k \cdot \mathbf{x} - \omega t) \]

where \( \mathbf{E}_0 \) is a constant vector is called a linearly polarized wave.

Consider

\[ \mathbf{E} = \mathbf{E} (\hat{x} \cos (k \cdot \mathbf{x} - \omega t) - \hat{y} \sin (k \cdot \mathbf{x} - \omega t)) \]
\[ \mathbf{B} = \frac{1}{c} \mathbf{k} \times \mathbf{E} (\hat{x} \cos (k \cdot \mathbf{x} - \omega t) - \hat{y} \sin (k \cdot \mathbf{x} - \omega t)) \]

For fixed \( x = 0 \)

\[ \mathbf{E} = \mathbf{E} (\hat{x} \cos \omega t + \hat{y} \sin \omega t) \]

In this case in a plane of fixed \( \hat{x} \), the direction of the field rotates.

This is called a circularly polarized wave.
In general a wave will be a superposition of waves with random polarizations. These are called unpolarized waves. Polarized sunglasses are transparent to waves that are linearly polarized in a direction.

Magnetic properties of matter recall in dielectric media

\[ \int \vec{E} \cdot \hat{n} \, dA = \frac{q}{\varepsilon_0} \rightarrow \frac{q}{\varepsilon} \]

\[ \omega = q_{\text{free}} + q_{\text{polariz}} \]

\[ \varepsilon = k \varepsilon_0 \]

\[ k = \text{dielectric constant} \]

For magnetic - in a medium with magnetic dipoles

\[ U = -\vec{\mu} \cdot \vec{B} \]

Dipoles lowest energy state is parallel to applied field

\[ \vec{B}_{\text{dip}} = \vec{\mu}_0 \left( \frac{3 \vec{r} (\vec{\mu} \cdot \vec{r}) - \vec{\mu} r^2}{r^5} \right) \]
The dipoles generate an additional magnetic field - you can think of dipoles like little bar magnets.

These tend to enhance the actual field.

From the expression \( \frac{B_{\text{dip}}}{\mu_0} = \frac{\text{dipole moment}}{\text{volume}} \) for the field due to a dipole,

\[ \vec{M} = \text{magnetization} = \frac{\text{dipole moment}}{\text{volume}} \]

\( \vec{B} = \text{actual magnetic field} \)

define \( \vec{H} \) by

\[ \vec{B} = \mu_0 \vec{H} \]

when there are no dipoles. When there are dipoles,

\[ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \]

here \( \vec{H} \) is like the applied field.
In some case

\[ M = \chi H \]

where \( \chi \) is called the magnetic susceptibility

\[ B = \mu_0 (H + \chi H) = \mu_0 (1 + \chi) H = \mu H \]

most materials have \( \chi > 0 \) (paramagnetic materials)

some have \( \chi < 0 \) (diamagnetic materials)

and some have a more complex \( H \) dependence (ferromagnetic)

\[ B = \mu_0 (H + \chi H) \]

\( x \) in paramagnetic materials have random dipoles that align with the fields.

\( x \) id diamagnetic substance - Lenz's law - with complications from quantum mechanics reduces the field strength. This effect is generally small.
The last case the dipoles are remain aligned for a long time because they are in a lower energy state. Eventually thermal effects take over when the applied field is turned off.

For permanent magnets the magnetization is not a single valued function of $H$.

\[ M \]

This phenomena is called hysteresis.
final remarks - in media

$\varepsilon_0 - \varepsilon \geq \varepsilon_0$

$\mu_0 - \mu \geq \mu_0$

$c^2 = \frac{1}{\varepsilon_0 \mu_0}$

$\frac{c^2}{n^2} = \frac{1}{\varepsilon_0 \mu_0}$

The speed of light in a medium with electric and magnetic dipole is

$\frac{c}{n}$

where

$n = (\frac{\varepsilon_0 \mu}{\varepsilon_0 \mu_0})^{1/2}$

is called the index of refraction.