Lecture 36

Special Relativity

Inertial coordinate systems

coordinate systems where free particles move with constant velocity

Galilean relativity - the laws of physics have the same form in all intertial coordinate systems.

The transformations that preserve Newtons second law without forces

$$m \frac{d^2\bar{r}}{dt^2} = 0$$

are

(1) $\bar{r} \to \bar{r}' = \bar{r} + \bar{r}_0$ $\quad \bar{r}_0$ constant

$\bar{r} \to \bar{r}' = \bar{r} + \bar{v}_0 t$ $\quad \bar{v}_0$ constant

$\bar{r} \to (\bar{r}')_i = \sum_j R_{ij} (\bar{r})_j$ $\quad$ (rotations)

$t \to t' = t + t_0$ $\quad$ time shifts maxwells equations without sources the waves move at the speed of light.

* this is not preserved under the translations $\bar{r} \to \bar{r}' = \bar{r} \pm \bar{V}_0 t$

questions : Is the earth is a special coordinate system? Michaelson-Morley experiment $\Rightarrow$ the speed of light is the same in moving coordinate systems

Since currents and charges are governed by Newtons laws, there is an inconsistency.

Einstein: Inertial coordinate systems are related by transformations that preserve Maxwells equations with no sources. they preserve

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \begin{Bmatrix} \bar{E}(\bar{r}t) \\ \bar{B}(\bar{r}t) \end{Bmatrix} = 0$$

It is useful to introduce some notation

(1) An "event" is something that happens at a given position $\vec{r}_A$ and time t_A in some coordinate system

(2) The space-time coordinates of an event are

$$x_A^\mu = (x_A^0\ x_A^1\ x_A^2\ x_A^3) = (ct_A, x_A\ y_A, z_A)$$

(3) The proper time $\Delta\tau_{AB}$ between events A and B is defined by

$$\Delta\tau_{AB}^2 = (t_A - t_B)^2 - \frac{1}{c^2}\left((x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2\right)$$

$$= \frac{1}{c^2}\left[(x_A - x_B)^{0\,2} - (x_A^1 - x_B^1)^2 - (x_A^2 - x_B^2)^2 - (x_A^3 - x_B^3)^2\right)$$

(4) We define the metric tensor

$$\eta_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2, \text{or } 3 \\ 0 & \mu \neq \nu \end{cases}$$

consequences

moving train

X = rest frame of station

X' = rest frame of train

event A

woman at train station lights match

event B

woman at train station blows out match

$$\Delta t_{AB} = t$$

$$\Delta x_{AB} = 0$$

to an observer in the train

$$\Delta t'_{AB} = t'$$

$$\Delta x'_{AB} = -vt'$$

$$c^2t^2 = c^2t'^2 - v^2t'^2 = (c^2-v^2)t'^2$$

$$t'^2 = \frac{c^2}{c^2-v^2}t^2 = \frac{1}{1-v^2/c^2}t^2$$

$$\boxed{t' = \sqrt{\frac{1}{1-v^2/c^2}}\,t} \qquad t' > t$$

$$\gamma \equiv \sqrt{\frac{1}{1-v^2/c^2}} > 1$$

This shows that the time elapsed in the moving frame is longer by a factor γ.

This is seen in muon decay - muons are unstable particles that are produced in the atmosphere

because when they reach the atmosphere they are traveling close to the speed of light their lifetime relative to a observer on earth is longer than for a muon at rest. -

more muons are seen a sea level than would be expected if there were no time dilation

Twin paradox

Twin in moving frames clock is slowed down - comes back younger

(This is no symmetric due to the acceleration needed to turn around - but the effect is observed in atomic clocks flown around the world)

Length of moving train

A front of train moves past observer
B back of train moves past observer

$$\Delta t_{AB} = \frac{L}{V} \qquad \Delta X_{AB} = 0$$

$$\Delta t'_{AB} = \frac{L'}{V} \qquad \Delta X'_{AB} = L'$$

$$c^2 \frac{L^2}{V^2} = c^2 \frac{L'^2}{V^2} - L'^2$$

$$L^2 = L'^2 \left(1 - \frac{V^2}{c^2}\right)$$

$$L = L' \sqrt{1 - V^2/c^2} = \frac{L'}{\gamma}$$

Train appears shorter to observer in station.

This phenomenon is called Lorentz contraction

The condition

$$(\bar{r}_A - \bar{r}_B)^2 - c^2 (t_A - t_B)^2 =$$
$$(\bar{r}'_A - \bar{r}'_B)^2 - c^2 (t'_A - t'_B)^2$$

can be solved. The solution is

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$$

where

$$a^{\mu} = (a^0, a^1, a^2, a^3) = \text{constant}$$

and $\Lambda^{\mu}{}_{\nu}$ satisfies

$$\eta_{\mu\nu} = \sum_{\alpha,\beta=0}^{3} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \eta_{\alpha\beta} \qquad *$$

where the numbers $\Lambda^{\mu}{}_{\nu}$ are constants

$\Lambda^{\mu}{}_{\nu}$ satisfying $*$ are called Lorentz transformations

special cases

① $ct' = ct + a^0$ $(\Lambda = 0)$ time translation

② $\vec{x}' = \vec{x} + \vec{a}$ $(\Lambda = 0)$ space translation

③ $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \left.\begin{matrix} ct \\ x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \\ z \end{matrix}\right\} \text{rotations}$$

$$(ct)' = \gamma(ct) + \gamma\left(\frac{v}{c}\right)x = \gamma\left(ct + \frac{v}{c}x\right)$$

$$x' = \gamma x + \gamma\left(\frac{v}{c}\right)(ct) = \gamma\left(x + \frac{v}{c}ct\right)$$

$$y' = y$$

$$z' = z$$

The inverse is obtained by changing the sign of v

$$c^2\Delta t'^2 - \Delta x'^2 = \gamma^2\left(c\Delta t + \frac{v}{c}\Delta x\right)^2 - \gamma^2\left(\Delta x + \frac{v}{c}c\Delta t\right)^2$$

$$= \frac{1}{1 - v^2/c^2}\left(c^2\Delta t^2 - v^2\Delta t^2 + \frac{v^2}{c^2}\Delta x^2 - \Delta x^2\right)$$

$$= \frac{1}{1 - v^2/c^2}\left(c^2\left(1 - \frac{v^2}{c^2}\right)\Delta t^2 - \left(1 - \frac{v^2}{c^2}\right)\Delta x^2\right)$$

$$= c^2\Delta t^2 - \Delta x^2$$

it is useful to define the dimensionless quantity $\beta = v/c$

It is constructive to analyze the train problem using Lorentz transformations

woman at station

$\Delta t_{AB} = t$

$\Delta x_{AB} = 0$

$c\Delta t'_{AB} = c\gamma \Delta t_{AB} + \beta\gamma(\Delta x_{AB}) = c\gamma\Delta t_{AB}$

$\Delta x_{AB'} = \gamma \Delta x_{AB} + \beta\gamma(c\Delta t_{AB}) = v\gamma\Delta t_{AB}$

$\boxed{\Delta t'_{AB} = \gamma \Delta t_{AB}}$

Length of train

$c\Delta t_{AB} = c\frac{L}{v}$

$\Delta x_{AB} = 0$

$c\Delta t'_{AB} = c\gamma \Delta t_{AB} + \beta\gamma(\Delta x_{AB}) =$

$\Delta x'_{AB} = \gamma \Delta x_{AB} + \beta\gamma(c\Delta t_{AB}) = \beta\gamma c\frac{L}{v} = \gamma L$

$\boxed{L = \frac{\Delta x'_{AB}}{\gamma}}$

Transformation properties of velocities

$$\frac{dx^{i'}}{dt} = \frac{dx^{i'}}{dt'}\frac{dt'}{dt}$$

$$\frac{dx^{0'}}{dt} = \frac{dx^{i}}{dt'}\frac{dt'}{dt}$$

recall

$$u_x = \frac{\Delta x}{\Delta t} \qquad u_x' = \frac{\Delta x'}{\Delta t'}$$

$$\Delta t' = \gamma(\Delta t' + \beta \tfrac{\Delta x}{c})$$

$$\Delta x' = \gamma(\Delta x + \beta \cdot c\Delta t)$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x + \beta c \Delta t}{\Delta t + \beta u_x/c} = \frac{u_x + v_x}{1 + \frac{u_x v_x}{c^2}} = u_x'$$

If we change the sign of v

$$u_x' = \frac{u_x - v_x}{1 + u_x v_x/c^2}$$

(This is for the parallel components —

for the $\perp$ components $\Delta x' = \Delta x$

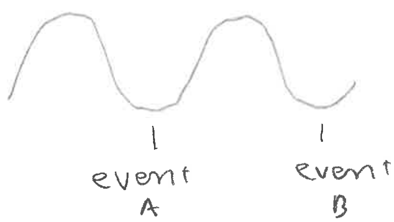
$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\gamma(\Delta t \quad)} \quad) = \frac{u_x}{\gamma}$$

$$u_{\parallel}' = \frac{u_{\parallel} + V}{1 + \frac{u_{\parallel}V}{c^2}} \qquad u_{\perp}' = \frac{1}{\gamma}u_{\perp}$$

Relativistic Doppler shift

$$\bar{E} = E_0 \cos(\bar{k}\cdot\bar{x} - \omega t)$$

changing coordinates at time will change x, t but it shold happen is a manner that preserves the phase

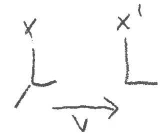


we expect

$$\bar{k}\cdot\bar{x} - \omega t = \bar{k}'\cdot\bar{x}' - \omega' t'$$

This will be true if $(\bar{k}, \omega)$ transform like $\bar{x}, t$

$$\begin{pmatrix} \frac{\omega}{c} \\ \bar{k} \end{pmatrix} \rightarrow \begin{matrix} \frac{\omega'}{c} = \gamma \frac{\omega}{c} \pm \gamma \frac{v}{c} k \\ k' = \gamma k \pm \gamma \frac{v}{c} \frac{\omega}{c} \end{matrix}$$



$x' = x - vt$
(lower sign)

in the direction of motion $\frac{\omega}{c} = k$

$$\frac{\omega'}{c} = \gamma (1 \pm \frac{v}{c}) \frac{\omega}{c}$$

$$\omega' = \frac{1}{\sqrt{1 - v^2/c^2}} = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\omega \quad a \quad \sqrt{\frac{1 + v/c}{1 - v/c}}\,\omega$$