Last Time

Special Relativity

4 vector coordinates of "events"

$\vec{r}_A, t_A \rightarrow x^\mu_A = (ct_A, x_A\ y_y\ z_A) = (x^0, x^1, x^2, x^3)$

metric tensor

$$\eta_{\mu\nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2 \text{ or } 3 \\ 0 & \mu \neq \nu \end{cases}$$

* The proper time or proper distance between events A and B is

$$c^2 \Delta \tau^2_{AB} = \sum_{\mu\nu=0}^{3} \eta_{\mu\nu} (x^\mu_A - x^\mu_B)(x^\nu_A - x^\nu_B) = -d^2_{AB}$$

$$= c^2 \Delta t^2_{AB} - |\vec{r}_A - \vec{r}_B|^2$$

* The proper time or proper distance between any 2 events is the same in all inertial coordinate systems.

This leads to

If x^μ are the 4 vector coordinates of an event in an inertial coordinate system then the 4 vector coordinates of the same event in any other inertial coordinate system are related by

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}} \qquad (1)$$

where

$$a^{\mu} = (a^0\ a^1\ a^2\ a^3)$$

is a constant 4 vector and $\Lambda^{\mu}{}_{\nu}$ are constants satisfying

$$\boxed{\eta_{\mu\nu} = \sum_{\alpha,\beta=0}^{3} \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}} \qquad (2)$$

The transformations (1) are called Poincaré transformations; the transformations $\Lambda^{\mu}{}_{\nu}$ satisfying (2) are called Lorentz transformations.

Poincare transformations

1. $x'^{\mu} = x^{\mu} + (a^0, 0, 0, 0)$

$$\Lambda^{\mu}{}_{\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

$a^i = 0 \quad i = 1,2,3$

(time translation)

2. $x'^{\mu} = x^{\mu} + (0, a^1, a^2, a^3)$

$a^0 = 0$

(space translation)

3. $x'^{\mu} = \Lambda^{\nu}{}_{\nu} x^{\nu} \qquad a^{\mu} = 0$

$\Lambda^0{}_0 = 1 \quad \Lambda^i{}_0 = \Lambda^0{}_i = 0$

$\Lambda^i{}_j = \text{rotation} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc.

4. rotationless Lorentz transformation

$x'^0 = \gamma x^0 \pm \gamma \frac{v}{c} x^1 \qquad a^{\mu} = 0$

$x'^1 = \gamma x^1 \pm \gamma \frac{v}{c} x^0$

etc.

$x'^2 = x^2$

$x_3'^1 = x^3$

Train example

① woman lighting and blows out match

$\Delta x = 0$

$\Delta t = t$

$\Delta x' = \gamma \Delta x - \gamma \frac{v}{c} \Delta t c$ (woman is behind train)

$c\Delta t' = c\gamma \Delta t - \gamma \frac{v}{c} \Delta x$

$\Delta t' = \gamma \Delta t$ since $\Delta x = 0$

Time between events is longer for observer on moving train

$\Delta x' = -\gamma v \Delta t$

this also means that according to an observer on the train the events do not take place at the same place

② event 1 - front of train passes woman, event 2 back of train passes woman

$\Delta x = 0$

$\Delta t = \frac{L}{v}$ L = Length of train according to woman

$$\Delta x' = \gamma\, \Delta x - \gamma \tfrac{v}{c}(c\Delta t)$$

$$c\Delta t' = \gamma (c\Delta t) - \gamma \left(\tfrac{v}{c}\right) \Delta x$$

$$\Delta x' = \gamma \tfrac{v}{c} c \left(\tfrac{L}{v}\right) = \gamma L$$

$L = \frac{1}{\gamma} L'$ (Lorentz contraction)

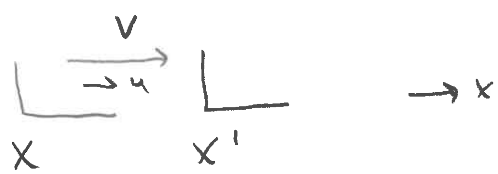
(train is shorter according to woman)

$$\Delta t' = \gamma \cdot \Delta t = \gamma \frac{L}{v}$$

(time between events according to train)

Given that we have Lorentz transformations we can look at other transformation

① velocity transformation



$$x' = \gamma x - \gamma \tfrac{v}{c} ct$$

$$ct' = \gamma ct - \gamma \tfrac{v}{c} x$$

$$dx' = \gamma\, dx - \gamma v\, dt$$

$$dt' = \gamma\, dt - \gamma \tfrac{v}{c^2}\, dx$$

$$u' = \frac{dx'}{dt'} = \frac{\gamma\, dx - \gamma v\, dt}{\gamma\, dt - \gamma \frac{v}{c^2} dx} = \frac{\gamma u - \gamma v}{\gamma\left(1 - \frac{uv}{c^2}\right)}$$

$$= \frac{u - v}{1 - \frac{uv}{c^2}}$$

If we change the sign of v

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}$$

as u v both approach $c \rightarrow \frac{2c}{1+1} = c$

the sum of both velocities is still c.

This is for the case where u and v are parallel

If u and v are perpendicular

$$ct' = \gamma ct$$

$$y' = y$$

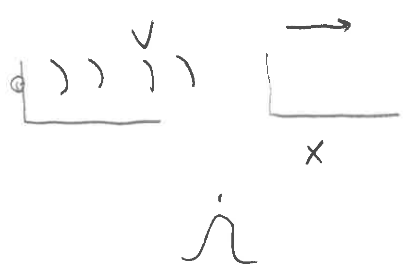
$$c\,dt' = \gamma c\,dt$$

$$dy' = dy$$

$$\frac{dy'}{dt'} = \frac{1}{\gamma}\frac{dy}{dt}$$

$$u_y' = \frac{1}{\gamma} u_y$$

Doppler shift



frequency in x' reduced

To treat this first note that if x^μ and y^μ are 4 vectors then

$$\sum \eta_{\mu\nu} x^\mu y^\nu = \sum \eta_{\mu\nu} x'^\mu y'^\nu$$

for $x'^\mu = \sum \Lambda^\mu{}_\alpha x^\alpha \quad y'^\nu = \sum \Lambda^\nu{}_\beta x^\beta$

$$\sum \underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}\, x^\alpha y^\beta$$

$$\sum \eta_{\alpha\beta} x^\alpha y^\beta$$

which proved the equation at the top of the page.

If both observers are looking at the tenth peak

$$\bar{E} = \bar{E}_0 \cos(\bar{k}\cdot\bar{r} - \omega t) \Rightarrow$$

$$\bar{k}\cdot\bar{r} - \omega t = \pi = \bar{k}'\cdot\bar{r}' - \omega' t'$$

$$\bar{k}\cdot\bar{r} - kct = \pi = \bar{k}'\cdot\bar{r}' - k'ct'$$

this will be invariant if

$$k^\mu = \begin{pmatrix} k \\ \vec{k} \end{pmatrix}$$

transforms like a 4 vector

$$k' = \gamma(k) - \gamma\frac{v}{c}k = \gamma\left(1-\frac{v}{c}\right)k$$

$$k' = \gamma(k) - \gamma\frac{v}{c}k = \gamma\left(1-\frac{v}{c}\right)k$$

note

$$\gamma\left(1-\frac{v}{c}\right) = \frac{1}{\sqrt{1-v^2/c^2}}\left(1-\frac{v}{c}\right) = \frac{\left(1-\frac{v}{c}\right)}{\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}} =$$

$$= \sqrt{\frac{1-v/c}{1+v/c}}$$

$$\boxed{\omega' = ck' = \sqrt{\frac{1-v/c}{1+v/c}}\cdot\omega}$$

for the wavelength $kL = 2\pi$ $L = 2\pi/k$

$$\boxed{L' = \sqrt{\frac{1+v/c}{1-v/c}}\,L}$$

If we reverse the direction the $v \to -v$ and the frequency increases.

Modification of Newtons second Law.

Let x^μ be the 4 vector coordinates of a particle

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu + a^\mu$$

consider a change in x^μ (a^μ does not change)

$$\Delta x^{\mu\prime} = \sum \Lambda^\mu{}_\nu \Delta x^\nu$$

divide $\Delta\tau$

$$\frac{\Delta x'^\mu}{\Delta\tau'} = \sum \Lambda^\mu{}_\nu \frac{\Delta x^\nu}{\Delta\tau'}$$

but $\Delta\tau$ = proper time $= \Delta\tau'$

$$\frac{\Delta x'^\mu}{\Delta\tau'} = \sum \Lambda^\mu{}_\nu \frac{\Delta x^\nu}{\Delta\tau}$$

If we take the limit $\Delta x^\mu, \Delta\tau \to 0$

$$\frac{dx'^\mu}{d\tau} = \sum \Lambda^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

$\frac{dx^\mu}{d\tau}$ is called the <u>4 velocity</u> –

it transforms like a 4 vector without the a^μ

a second derivative gives

$$\frac{d^2x^\mu}{d\tau^2} = 4 \text{ acceleration}$$

recall

$$d\tau^2 = dt^2 - \frac{1}{c^2}dx^2$$

$$= \left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)dt^2$$

$$= \frac{1}{\gamma^2}\, dt^2$$

$$\boxed{\frac{dt}{d\tau} = \gamma}$$

$$\boxed{\frac{dx^\mu}{d\tau} = \left(c\gamma,\ \frac{d\bar{x}}{dt}\cdot\frac{dt}{d\tau}\right) = (c\gamma,\ \bar{v}\gamma)}$$

if we differentiate again

$$\frac{d\bar{v}}{dt}\cdot\frac{dt}{d\tau} = \bar{a}\cdot\gamma$$

$$\frac{d^2x^\mu}{d\tau^2} = \left(c\,\frac{d\gamma}{d\tau},\ \frac{d\bar{v}}{d\tau}\gamma + \bar{v}\,\frac{d\gamma}{d\tau}\right)$$

$$\frac{d\gamma}{d\tau} = \frac{d}{d\tau}\frac{1}{\sqrt{1-v^2/c^2}} = \left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\left(-\frac{2v\cdot dv}{c^2\, d\tau}\right)$$

$$= \left(1-\frac{v^2}{c^2}\right)^{-3/2}\frac{v}{c^2}\cdot\frac{dv}{dt}\cdot\frac{dt}{d\tau}$$

$$= \gamma^3\,\frac{v}{c^2}\cdot\gamma\,\frac{d\bar{v}}{dt}$$

$$= \gamma^4\,\frac{1}{c^2}\,\bar{v}\cdot\frac{d\bar{v}}{dt}$$

$$= \gamma^4\,\frac{1}{c^2}\,\bar{v}\cdot\bar{a}$$

In this case the 4 acceleration can be expressed as

$$\frac{d^2x^\mu}{d\tau^2} = \left(\gamma^3 \frac{1}{c}\, \bar{v}\cdot\bar{a},\ \bar{a}\cdot\gamma^2 + \bar{v}\cdot\gamma^4 \frac{1}{c^2}\, \bar{v}\cdot\bar{a}\right)$$

for $\bar{v}=0$ (rest frame of particle

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1$$

$$\frac{d^2x^\mu}{d\tau^2} \rightarrow (0, \bar{a}) = \left(0, \frac{d^2v}{dt^2}\right)$$

If we assume newtons second law holds in the particles instanteous rest frame

$$\frac{d^2x^\mu}{d\tau^2} \Rightarrow (0, \bar{a}) = \left(0, \frac{1}{m}\bar{F}\right)$$

or $\quad 0 = \left(0, \bar{a} - \frac{1}{m}\bar{F}\right) = 0$

If a 4 vector is 0 in one inertial coordinate system then it is 0 in <u>all</u> inertial coordinate systems. (if there is no a^μ)

This suggests that if we choose

$\frac{1}{m} f^{\mu}$ to be a 4 vector that

becomes $\begin{pmatrix} 0 \\ \bar{F}/m \end{pmatrix}$ in the particles

rest frame, then we have

$$\boxed{m \frac{d^2 x^{\mu}}{d\tau^2} = f^{\mu}}$$

To find f^{μ} in a coordinate system where the particle has velocity V in some direction

$$f^0 = \gamma \frac{\bar{V} \cdot \bar{F}}{c}$$

$$\bar{f}_{\parallel} = \gamma \bar{F}$$

In this way

(1) newtons second law has the expected form in the particles instantaneous rest frame

(2) $f^{\mu} = (0 \ \bar{F})$ in rest frame, it transforms like a 4 vector with no a^{μ}

If the force is 0 then $f^\mu = 0$
and we have

$$m \frac{d^2x^\mu}{d\tau^2} = 0$$

or

$$m \frac{dx^\mu}{d\tau} = p^\mu \quad \text{is conserved}$$

p^μ is called the 4 momentum

$$p^\mu = m\frac{dx^\mu}{d\tau} = \left(mc\frac{dt}{d\tau},\ m\frac{dx}{dt}\frac{dt}{d\tau}\right)$$
$$= (mc\gamma,\ m\bar{v}\gamma)$$

$m\gamma\bar{v}$ is called the relativistic momentum.

$$= \frac{mv}{\sqrt{1-v^2/c^2}}$$

it is conserved in the absence of forces

$$p^0 = mc\gamma = mc\frac{1}{\sqrt{1-v^2/c^2}} \approx mc\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{1}{8}\left(\frac{v^2}{c^2}\right)^2\ \right)$$
$$= mc + \tfrac{1}{2}mv^2/c + \cdots$$

$$p^0c = mc^2 + \tfrac{1}{2}mv^2 + \cdots$$

we see that this conserved quantity looks like a constant, + the kinetic energy + terms that vanish as $c \to \infty$

This is interpreted as the relativistic energy

when $v=0$ the energy becomes mc^2

$$E^2 = m^2c^4\left(\frac{1}{1-v^2/c^2}\right)$$

$$E^2 = m^2c^4 + \frac{E^2v^2}{c^2} \qquad \frac{E^2v^2}{c^2} = \frac{(\gamma mc^2)^2 v^2}{c^2}$$

$$\boxed{E^2 = m^2c^4 + p^2c^2} \qquad = c^2\gamma^2m^2v^2$$

$$= c^2p^2$$

where p is the relativistic momentum