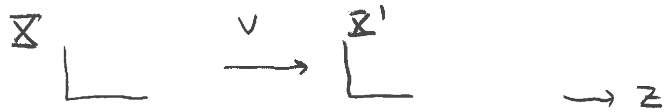
Lecture 37

Adding velocities

Consider



Let Δz and Δt be small displacements in coordinate system Σ

In x' these are

$$c\Delta t' = \gamma c \Delta t - \gamma \beta \Delta z$$

$$\Delta z' = \gamma \Delta z - \gamma \beta c \Delta t$$

dividing $\Delta z'$ by $c\Delta t'$

$$\frac{\Delta z'}{c\Delta t'} = \frac{\gamma \Delta z - \gamma \beta c \Delta t}{\gamma c \Delta t - \gamma \beta \Delta z} = \frac{\Delta z - \beta c \Delta t}{c(\Delta t - \beta \Delta z / c)}$$

$$\frac{\Delta z'}{\Delta t'} = \frac{(\Delta z / \Delta t) - \beta c}{1 - \beta \left(\frac{\Delta z}{\Delta t}\right) \cdot \frac{1}{c}}$$

$$\beta c = \frac{v}{c} c = v$$

as $\Delta t \quad \Delta z \longrightarrow 0 \qquad \frac{\Delta z}{\Delta t} \longrightarrow u_z \qquad \frac{\Delta z'}{\Delta t'} = u_z'$

$$\therefore \quad \boxed{u_z' = \frac{u_z - v}{1 - u_z v / c^2}}$$

This corresponds to a positive velocity in the z direction in X -
If we go in the opposite direction the becomes

$$\boxed{u_z' = \frac{u_z + V}{1 + u_z V/c^2}} \qquad \bar{u} \parallel \bar{v}$$

If both u_z and v approach c

$$u_z \to \frac{c+c}{1+c^2/c^2} = c$$

we see that the speed of light is the same in both coordinate systems

Next consider the case that the velocity is $\perp$ to V - In that case we take $\Delta z = 0$

$$c\Delta t' = \gamma c \Delta t - \gamma\beta\, 0$$

$$\Delta z' = 0 - \gamma\beta\, c\Delta t$$

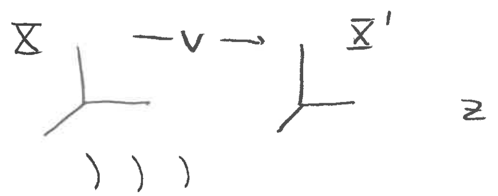
$$\Delta x' = \Delta x$$

$$\frac{1}{c}\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\gamma c \Delta t}$$

Taking limits as $\Delta x, \Delta t \to 0$ gives

$$\boxed{u_x' = \frac{1}{\gamma} u_x \qquad \bar{u} \perp \bar{v}}$$

relativistic doppler shift



consider an electromagnetic field

$$\bar{E}(rt) = \bar{E}_0 \cos(\bar{k}\cdot\bar{r} - \omega t)$$

$$\bar{E}'(rt) = \bar{E}_0 \cos(\bar{k}'\cdot\bar{r}' - \omega' t')$$

consider the 10 maximum at same $\bar{r}$ at t in system X – it will be at $\bar{r}'$ t' in system X'

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}$$

the peak could be determined by a light that blinks at $\bar{r}, t$.

we must have the same phase

$$\bar{k}\cdot\bar{r} - \omega t = \bar{k}'\cdot\bar{r}' - \omega' t' = 10\pi$$

note that if

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}$$

$$y'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} y^{\nu}$$

$$\sum_{\mu\nu} \eta_{\mu\nu} x'^{\mu} y'^{\nu} = \sum_{\mu\nu\alpha\beta} \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} x^{\alpha} \Lambda^{\nu}{}_{\beta} y^{\beta}$$

$$= \sum_{\alpha\beta} \left(\sum_{\mu\nu} \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \right) x^{\alpha} y^{\beta}$$

$$= \sum_{\alpha\beta} \eta_{\alpha\beta} x^{\alpha} y^{\beta}$$

This shows that $\sum_{\mu\nu} \eta_{\mu\nu} x^{\mu} y^{\nu}$ has the same value in all inertial coordinate systems

This means that if we want to keep the phase constant we want

$$k^{\mu} = \left(\frac{\omega}{c}, \bar{k}\right)$$

$$k'^{\mu} = \Lambda^{\mu}{}_{\nu} k^{\nu}$$

Then

$$k \cdot \bar{r} - \omega t = \bar{k}' \cdot \bar{r}' - \omega' t'$$

$$\frac{\omega'}{c} = \gamma \frac{\omega}{c} - \beta\gamma k_z$$

$$k_z' = \gamma k_z - \beta\gamma \frac{\omega}{c}$$

since $\omega = ck$
$\bar{k} = (0\ 0\ k)$

$$\omega' = \gamma\omega - c\beta\gamma k$$

$$ck' = ck\gamma - vk\gamma$$

$$k' = \gamma\left(1 - \frac{v}{c}\right)k = \frac{1-\beta}{\sqrt{1-\beta^2}}\,k = \frac{1-\beta}{\sqrt{(1-\beta)(1+\beta)}}\,k$$

$$= \sqrt{\frac{1-\beta}{1+\beta}}\,k = \sqrt{\frac{c-v}{c+v}}\,k$$

since $\omega = ck$ $\quad \lambda = \frac{2\pi}{k}$

$$\boxed{\omega' = \sqrt{\frac{c-v}{c+v}}\,\omega} \quad \boxed{\lambda' = \sqrt{\frac{c+v}{c-v}}\,\lambda}$$

away from source

This is for x' moving away from the source of radiation. If x' is moving towards the source the sign of v is reversed

$$\boxed{\omega' = \sqrt{\frac{c+v}{c-v}}\,\omega} \quad \boxed{\lambda' = \sqrt{\frac{c-v}{c+v}}\,\lambda}$$

towards source

Note if $v \ll c$

$$\omega' = \sqrt{\frac{1-v/c}{1+v/c}}\,\omega \cong \left(1 - \frac{1}{2}\frac{v}{c} - \frac{1}{2}\frac{v}{c}\right)\omega = \left(1 - \frac{v}{c}\right)\omega$$

It is inconsistent to have

(1) Newtons Second Law

(2) Maxwells equations

(3) Inertial coordinate systems where the laws of physics have the same form

Michaelson Morley experiment $\Rightarrow$ Newton's second law needs to be modified to be made consistent with maxwells equations

consider x^μ = 4 vector coordinates of a particle

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu + a^\mu$$

let

$$\Delta x^\mu = x_A^\mu - x_B^\mu$$

$$\Delta x'^\mu_{AB} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \Delta x^\nu_{AB}$$

divide both sides by $\Delta \tau_{AB}$

$$\frac{\Delta x'^\mu_{AB}}{\Delta \tau'_{AB}} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \frac{\Delta x^\nu_{AB}}{\Delta \tau'_{AB}}$$

since $\Delta \tau_{AB} = \Delta \tau_{AB}'$ this becomes

$$\frac{\Delta x_{AB}^{\mu\prime}}{\Delta \tau_{AB}'} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} \frac{\Delta x_{AB}^{\nu}}{\Delta \tau_{AB}}$$

If we take the limit $\Delta x, \Delta \tau \to 0$ we get

$$\boxed{\frac{dx^{\mu\prime}}{d\tau} = \sum \Lambda^{\mu}{}_{\nu} \frac{dx^{\nu}}{d\tau}}$$

This means that if x^{μ} transforms like a 4 vector so does $\frac{dx^{\mu}}{d\tau}$, except without the factor a^{μ}

$$\boxed{\frac{dx^{\mu}}{d\tau} = \text{called the 4 velocity}}$$

Note

$$(\Delta \tau)^2 = (\Delta t)^2 - \frac{1}{c^2}(\Delta x)^2$$

$$\left(\frac{\Delta t}{\Delta \tau}\right)^2 = 1 + \frac{1}{c^2}\left(\frac{\Delta x}{\Delta \tau}\right)^2 = 1 + \frac{1}{c^2}\left(\frac{\Delta x}{\Delta t}\right)^2\left(\frac{\Delta t}{\Delta \tau}\right)^2$$

This gives

$$1 = \left(1 - \left(\frac{\Delta x}{\Delta t}\right)^2 \frac{1}{c^2}\right)\left(\frac{\Delta t}{\Delta \tau}\right)^2$$

Taking limits $\Delta x, \Delta t\ \Delta \tau \to 0$

$$1 = \left(1 - \frac{v^2}{c^2}\right)\left(\frac{dt}{d\tau}\right)^2$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

$$\therefore \quad \boxed{\frac{dt}{d\tau} = \gamma}$$

using this the 4 velocity becoms

$$\frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma\,(c, \bar{v})$$

since $\frac{dx^\mu}{d\tau}$ is already a 4 vector, if we replace x'^μ by $\frac{dx^\mu}{d\tau}$ we get

$$\boxed{\begin{array}{l} \dfrac{d^2 x'^\mu}{d\tau^2} = \sum_\nu \Lambda^\mu{}_\nu \dfrac{d^2 x^\nu}{d\tau} \\[2ex] \dfrac{d^2 x^\mu}{d\tau^2} = 4 \text{ acceleration} \end{array}}$$

It follows that

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dx^\mu}{d\tau}\right) =$$

$$= \gamma \frac{d}{dt}\left(\gamma(c, v)\right)$$

$$= \gamma \frac{d\gamma}{dt}(c, v) + \gamma^2 \left(0, \frac{d\bar{v}}{dt}\right)$$

we note

$$\frac{d\gamma}{dt} = \frac{d}{dt} \frac{1}{\sqrt{1 - v^2/c^2}} = \left(-\frac{1}{2}\right) \frac{1}{(1 - v^2/c^2)^{3/2}} \left(-2v \frac{dv}{dt} \cdot \frac{1}{c^2}\right)$$

$$= \frac{1}{c^2} \gamma^3 \, \bar{v} \cdot \bar{a} \qquad \bar{a} = \frac{d\bar{v}}{dt} = \text{acceleration}$$

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{c^2} \gamma^4 (\bar{v} \cdot \bar{a})(c, \bar{v}) + \gamma^2 (0, \bar{a})}$$

when $\bar{v} = 0$ $\quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \to 0$ $\quad$ and

$$\frac{d^2 x^\mu}{d\tau^2} \to \left(0, \frac{d^2 \bar{r}}{dt^2}\right) = (0, \bar{a}) \qquad (\bar{v} = 0 \text{ coord system})$$

$$\boxed{m \frac{d^2 x^\mu}{d\tau^2} = (0, m\bar{a}) \qquad \text{in coord system where } v = 0}$$

To make Newtons second law compatible with maxwells equations and be valid for speed $<<< c$ we assume

① $m \frac{d^2 x^\mu}{d\tau^2} = f^\mu$

$f^\mu = 4$ vector force

② $f^\mu = (0, \bar{F})$ in instantaneous rest frame

To find f^μ in a coordinate system where the particle has velocity $\bar{v}$ we use lorentz transformation

$$f^\mu = \sum \Lambda^\mu{}_\nu (0\ \bar{F})$$

$$= \begin{pmatrix} \gamma \bar{\beta}\cdot\bar{F} \\ F_\perp \\ \gamma \bar{F}_\parallel \end{pmatrix}$$

$$\boxed{\begin{aligned} f^0 &= \gamma \frac{\bar{v}\cdot\bar{F}}{c} \\ \vec{f} &= \bar{F} + (\gamma - 1)\frac{\bar{v}\cdot\bar{F}}{\bar{v}\cdot\bar{v}}\bar{v} \end{aligned}}$$

$$\boxed{\begin{aligned} m\frac{d^2 x^0}{d\tau^2} &= \gamma \frac{\bar{v}\cdot\bar{F}}{c} \\ m\frac{d^2 \bar{x}}{d\tau^2} &= \bar{F} + (\gamma - 1)\frac{\bar{v}\cdot\bar{F}}{\bar{v}\cdot\bar{v}}\bar{v} \end{aligned}}$$

Relativistic Form of Newton's second Law

Note that when $\bar{F} = 0$ $f^{\mu} = 0$
this means

$$m \frac{d^2 x^{\mu}}{d\tau^2} = 0$$

or in the absence of external forces

$$\boxed{m \frac{dx^{\mu}}{d\tau} = \text{constant}}$$

$$p^{\mu} \equiv m \frac{dx^{\mu}}{d\tau} = m \left(\frac{dx^0}{d\tau}, \frac{d\bar{x}}{d\tau} \right)$$

$$= m\gamma \left(\frac{dx^0}{dt}, \frac{d\bar{x}}{dt} \right)$$

$$= m\gamma (c, \bar{v})$$

is called the 4 momentum

* momentum conservation
is replaced by

$$\bar{p} = m\gamma\bar{v} \quad \text{is conserved}$$

* $$m\gamma c = \frac{mc}{\sqrt{1 - v^2/c^2}} \simeq mc \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{1}{8}\left(\frac{v^2}{c^2}\right)^2 + \ldots \right)$$

$$m\gamma c^2 = mc^2 + \underbrace{\frac{1}{2}mv^2}_{\text{Non relativistic energy}} + \frac{c}{8}\left(\frac{v^2}{c^2}\right)^2 + \ldots$$

we identify

$$\boxed{\begin{aligned} E = m\gamma c^2 &= \text{relativistic conserved energy} \\ mc^2 &\equiv \text{rest energy} \end{aligned}}$$

note

$$\eta_{\mu\nu} p^\mu p^\nu = m^2\gamma^2 c^2 - m^2\gamma^2 v^2$$

$$= m^2\left(\frac{1}{1-v^2/c^2}\right)\cdot\left(1-v^2/c^2\right) c^2$$

$$= m^2 c^2$$

we can express this as

$$m^2 c^4 = E^2 - p^2 c^2$$

$$\boxed{E^2 = m^2 c^4 + p^2 c^2}$$

what about maxwells equations

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

this is called the electromagnetic field strength tensor.

consider

$$① \quad \sum_{\mu=0}^{3} \frac{\partial F^{\mu\nu}}{\partial x^{\mu}}$$

$\nu = 0$

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} =$$

$$-\frac{1}{c}\bar{\nabla}\cdot E = -\frac{1}{c}\frac{\rho}{\epsilon_0} = -\frac{c}{c^2\epsilon_0}\rho = -c\mu_0\rho$$

$\nu = 1$

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} =$$

$$\frac{\partial E_x}{\partial t}\frac{1}{c^2} + 0 - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} =$$

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} - (\nabla \times B)_x = -\mu_0 J_x$$

$$(\nabla \times B)_x = \mu_0 \left(J_x + \epsilon_0 \frac{\partial E}{\partial t}\right)$$

This continues for y, z

$$(\bar{\nabla} \times \bar{B}) = \mu_0 \left(\bar{J} + \epsilon_0 \frac{\partial E}{\partial t}\right)$$

we define $J^{\mu}(x) = (c\rho, \bar{J})$

$$\boxed{\sum_{\mu=0}^{3} \frac{\partial F^{\mu\nu}}{\partial x^{\mu}} = -\mu_0 J^{\mu}}$$

(includes Gauss' and Amperes Law)

The other equations have a similar form.

we define

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$$

$$\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \qquad \varepsilon^{0123} = \begin{cases} 1 \\ \text{completely antisymmetri} \end{cases}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\partial_\mu \tilde{F}^{\mu\nu}$$

$$\nu=0 \qquad \partial_0\tilde{F}^{00} + \partial_1\tilde{F}^{10} + \partial_2\tilde{F}^{20} + \partial_3 F^{30} = -\bar{\nabla}\cdot\bar{B} = 0$$

$$\nu=1 \qquad \partial_0\tilde{F}^{01} + \partial_1\tilde{F}^{11} + \partial_2\tilde{F}^{21} + \partial_3\tilde{F}^{31}$$

$$\frac{1}{c}\frac{\partial B_x}{\partial t} + 0 + \frac{\partial E_z}{\partial y}\frac{1}{c} - \frac{\partial E_y}{\partial z}\frac{1}{c}$$

$$\frac{1}{c}\left(\frac{\partial B}{\partial t} + \bar{\nabla}\times\bar{E}\right)_x = 0$$

In general

$$\bar{\nabla}\cdot\bar{B} = 0 \qquad \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$$

This show that Maxwells equations become

$$\boxed{\begin{array}{ll} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu & \text{Gauss / Ampere} \\ \partial_\mu \tilde{F}^{\mu\nu} = 0 & \text{Faraday / Gauss Mag} \end{array}}$$

Transformation properties of fields

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = \sum_{\alpha\beta} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \qquad (1)$$

$$\tilde{F}^{\mu\nu} \rightarrow \tilde{F}'^{\mu\nu} = \sum_{\alpha\beta} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tilde{F}^{\alpha\beta} \qquad (2)$$

note both $\tilde{F}$ and F have the same information. Quantities that transform like (1) (2) are called rank 2 tensors

It is possible to show that the equations at the top of the page have the same form in any inertial coordinate system consider

$$\frac{\partial}{\partial x^{\mu'}} F'^{\mu\nu} = \frac{\partial}{\partial x^{\mu'}} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

$$= \left(\Lambda^{\mu}{}_{\alpha} \frac{\partial}{\partial x^{\mu'}}\right) \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

note

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^{\rho}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\rho}} \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \rightarrow \Lambda^{\mu}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$$

$$= \sum_{\mu\rho} \left(\frac{\partial x^{\mu'}}{\partial x^{\alpha}} \cdot \frac{\partial x^{\rho}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\rho}} \right) = \frac{\partial}{\partial x^{\alpha}}$$

$$\therefore \quad \frac{\partial F'^{\mu\nu}}{\partial x^{\mu'}} = \Lambda^{\nu}{}_{\beta} \frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \Lambda^{\nu}{}_{\beta} \left(-\mu_0 J^{\beta}\right) = -\mu_0 J'^{\beta}$$

similary,

$$\frac{\partial^2 \tilde{F}'^{\mu\nu}}{\partial x^{\mu'}} = \Lambda^{\nu}{}_{\alpha} \frac{\partial \tilde{F}^{\mu\alpha}}{\partial x^{\mu}} = 0$$

This shows that as long as $(c\rho, \vec{J})$ transforms as a 4 vector, maxwells equations have the same form in all inertial coordinate systems