Relativity and Maxwells equations

① Special Relativity

The Laws of physics have the same form in any inertial coordinate system

② The proper time or proper distance between any two events is the same in any inertial coordinate systems

$$c^2 \Delta \tau^2 = c^2 \Delta t^2 - (\Delta \vec{x})^2$$
$$= c^2 (\Delta t')^2 - (\Delta \vec{x}')^2$$

③ The most general transformations that preserve the proper time between any 2 events have the form

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$$

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu + a^\mu$$

a^μ constant $\quad \Lambda^\mu{}_\nu$ constant

$$\eta_{\mu\nu} = \sum_{\alpha\beta} \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$$

④ $\eta_{\mu\nu} = \begin{cases} 1 & \mu=\nu=0 \\ -1 & \mu=\nu=-1 \\ 0 & \mu\neq\nu \end{cases}$

Differential Form of Maxwells Equations

$$\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$

$$\bar{\nabla}\times\bar{B} = \frac{1}{c^2}\frac{\partial E}{\partial t} + \mu_0\bar{J}$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Definition: Lorentz Tensors – these are objects that transform like products of 4 vectors under Lorentz transformation

$$\boxed{T^{\mu\nu} \to T^{\mu\nu'} = \sum_{\alpha\beta} \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}}$$

we define

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_y}{c} & 0 \end{pmatrix}$$

consider

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} \quad \text{and} \quad \sum_{\mu=0}^{3} \partial_\nu \tilde{F}^{\mu\nu}$$

case 1 $\nu = 0$

$$\frac{1}{c}\frac{\partial}{\partial t} F^{00} + \frac{\partial}{\partial x} F^{10} + \frac{\partial}{\partial y} F^{20} + \frac{\partial}{\partial z} F^{30} =$$

$$0 - \frac{1}{c}\left(\frac{\partial E_y}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = -\frac{1}{c}\bar{\nabla}\cdot \bar{E} = -\frac{1}{c\epsilon_0}\rho$$

$$= -\frac{c}{c^2\epsilon_0}\rho = -c\frac{\mu_0\epsilon_0}{\epsilon_0} = -\mu_0 c\rho$$

$$\partial_\mu F^{\mu 0} = -\mu_0 c\rho \qquad \text{(Gauss' Law)}$$

similarly

$$\partial_\mu \tilde{F}^{\mu 0} = \frac{1}{c}\frac{\partial}{\partial t}\tilde{F}^{00} + \frac{\partial}{\partial x}\tilde{F}^{10} + \frac{\partial}{\partial y}\tilde{F}^{20} + \frac{\partial}{\partial z}\tilde{F}^{30}$$

$$= -\vec{\nabla}\cdot B = 0$$

which gives

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \text{(Gauss Law for magnets)}$$

case 2 $\nu = 1$

$$\sum_\mu \frac{\partial F^{\mu 1}}{\partial x^\mu} = \frac{1}{c}\frac{\partial}{\partial t}F^{01} + \frac{\partial}{\partial x}F^{11} + \frac{\partial}{\partial y}F^{21} + \frac{\partial}{\partial z}F^{31}$$

$$= \frac{1}{c^2}\frac{\partial E_x}{\partial t} + 0 - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}$$

$$= \frac{1}{c^2}\frac{\partial E_x}{\partial t} - (\nabla\times B)_x = -\mu_0 \bar{J} \qquad \text{(Ampere)}$$

for $\tilde{F}$ we replace $\frac{E}{c} \to B$ $\quad B \to -\frac{E}{c}$

$$\frac{1}{c}\frac{\partial B_x}{\partial t} - \left(\nabla\times\left(-\frac{E}{c}\right)\right) = \frac{1}{c}\left(\frac{\partial B}{\partial t} + \nabla\times E\right) = 0 \qquad \text{(Faraday)}$$

In this language Maxwells equations have the form

$$\boxed{\begin{aligned} &\sum \partial_\mu F^{\mu\nu} = -\mu_0 (c\rho, \bar{J}) \\ &\sum \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{aligned}}$$

Next we show - If $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ transform like 2nd rank tensors as

$$J^\mu = (\rho c, \vec{J})$$

transforms like a 4 vector, then if

$$\sum \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$$

$$\sum \partial_\mu \tilde{F}^{\mu\nu} = 0$$

then they hold in all inertial coordinate systems.

Note

$$\frac{\partial}{\partial x'^\mu} = \sum_\nu \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$$

consider:

$$\frac{\partial}{\partial x'^\mu} F'^{\mu\nu} = \sum \frac{\partial x^\alpha}{\partial x'^\mu} \times \frac{\partial}{\partial x^\alpha} \Lambda^\mu{}_\beta \Lambda^\nu{}_\rho F^{\beta\rho} -$$

note

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \sum \Lambda^\mu{}_\nu \delta^\nu_\alpha + 0 = \Lambda^\mu{}_\alpha$$

$$\frac{\partial}{\partial x'^{\mu}} F'^{\mu\nu} = \sum \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \cdot \frac{\partial x'^{\mu}}{\partial x^{\beta}} \lambda^{\nu}{}_{\rho} F^{\beta\rho} =$$

$$\sum \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\beta}}}_{\delta^{\alpha}{}_{\beta}} \frac{\partial}{\partial x^{\alpha}} \cdot \lambda^{\nu}{}_{\rho} F^{\beta\rho} =$$

$$\sum \lambda^{\nu}{}_{\rho} \frac{\partial F^{\beta\rho}}{\partial x^{\beta}} = \sum \lambda^{\nu}{}_{\rho}(-\mu_0 J^{\rho}) = -\mu_0 J'^{\rho}$$

assuming $J^{\rho}(x)$ transforms like a 4 vector

$$\therefore \quad \sum \partial_{\mu} F^{\mu\nu} = -\mu_0 J^{\nu} \Rightarrow$$

$$\sum \partial'_{\mu} F'^{\mu\nu} = \sum \lambda^{\nu}{}_{\rho} (J^{\rho})(-\mu_0)$$

the algebra is identical for $\tilde{F}$

$$\sum \partial'_{\mu} \tilde{F}'^{\mu\nu} = \sum \lambda^{\nu}{}_{\alpha} (\partial_{\mu} \tilde{F}^{\mu\alpha}) = 0$$

these equations mean that if $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ transform like tensors and $J^{\mu} = (\rho c, \vec{J})$ transforms like a 4 vector, then maxwells equations have the same form in any inertial coordinate system

Fields in moving coordinate systems

$$\Sigma \xrightarrow{\quad v \quad} \Sigma'$$

$$z' = \gamma z - (ct)\gamma\beta$$

$$ct' = \gamma ct - \gamma\beta z$$

$$x' = x \;,\; y' = y$$

$$\begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

Assume we have an electric field in the z direction in Σ

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & 0 & 0 \\ -\frac{E_x}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F'^{\mu\nu} = \Lambda^\mu{}_1 \Lambda^\nu{}_1 \cancel{F^{11}}$$

$$\Lambda^\mu{}_1 \Lambda^\nu{}_0 F^{10}$$

$$\Lambda^\mu{}_0 \Lambda^\nu{}_1 F^{01}$$

$$\Lambda^\mu{}_0 \Lambda^\nu{}_0 \cancel{F^{00}}$$

$$= \Lambda^\mu{}_1 \Lambda^\nu{}_0 \left(-\frac{E}{c}x\right) + \Lambda^\mu{}_0 \Lambda^\nu{}_1 \left(\frac{E}{c}x\right)$$

in this case

$$F'^{01} = \frac{E_x'}{c} = \Lambda^0{}_1 \Lambda^1{}_0 \left(-\frac{E_x}{c}\right) + (\Lambda^0{}_0 \Lambda^1{}_1)\left(\frac{E_x}{c}\right)$$

$$= 0 \cdot 0 \cdot \left(-\frac{E_y}{c}\right) + \gamma \cdot 1 \cdot \frac{E_y}{c}$$

this gives

$$E_x' = \gamma E_x$$

Next consider E_z'

$$\frac{E_z'}{c} = F'^{03} = \Lambda^0{}_1 \Lambda^3{}_0 F^{10} + \Lambda^0{}_0 \Lambda^3{}_1 F^{01}$$

$$0 \cdot \gamma\beta \cdot F^{10} + \gamma \cdot 0 \quad F^{01}$$

$$E_z' = 0$$

$$B_y' = F^{31} = \underbrace{\Lambda^3{}_1}_{0} \underbrace{\Lambda^1{}_0}_{0} \left(-\frac{E_x}{c}\right) + \underbrace{\Lambda^3{}_0 \Lambda^1{}_1}_{\gamma\beta \cdot 1} \left(\frac{E_x}{c}\right)$$

$$B_y' = \gamma\beta \frac{E_x}{c}$$

$$B_x' = F^{23} = \underbrace{\Lambda^2{}_1}_{0} \underbrace{\Lambda^3{}_0}_{\gamma\beta} \left(-\frac{E_x}{c}\right) + \underbrace{\Lambda^2{}_0}_{0} \underbrace{\Lambda^3{}_1}_{0} \left(\frac{E_x}{c}\right)$$

$$= 0$$

$$B_z' = F^{12} = \Lambda^1{}_1 \Lambda^2{}_0 \left(-\frac{E_y}{c}\right) + \Lambda^1{}_0 \Lambda^2{}_1 \left(\frac{E_y}{c}\right)$$

$$\gamma \quad 0 \qquad\qquad 0 \quad 0$$

$$B_z' = 0$$

Note

$$E_y' = \gamma E_y$$

$$B_y' = \gamma \beta \frac{E}{c}$$

$$\frac{B_y'}{E_y'} = \beta \frac{1}{c}$$

This becomes $\frac{1}{c}$ in the limit $\beta \to 1$ $(v \to c)$

What happens if E is parallel to v

$$E = (0\ 0\ E_z)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{E_z}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{E_z}{c} & 0 & 0 & 0 \end{pmatrix}$$

$$F^{\mu\nu\prime} = \Lambda^\mu{}_0 \Lambda^\nu{}_3 \frac{E_z}{c} + \Lambda^\mu{}_3 \Lambda^\nu{}_0 \left(-\frac{E_z}{c}\right)$$

$$E'^{03} = \Lambda^0{}_0 \Lambda^3{}_3 \frac{E_z}{c} + \Lambda^0{}_3 \Lambda^3{}_0 \left(-\frac{E_z}{c}\right)$$

$$= \gamma^2 \frac{E_z}{c} + \gamma^2 \beta^2 \left(-\frac{E_z}{c}\right)$$

$$= \gamma^2 (1-\beta^2) \frac{E_z}{c}$$

$$= \frac{E_z}{c}$$

In this case the field does not change

$$F^{ij} = (\Lambda^i{}_0 \Lambda^j{}_3 - \Lambda^i{}_3 \Lambda^j{}_0)\left(\frac{E_z}{c}\right) = 0$$

$(i=3,\ j=3)$ $(i=3)(j=3)$

for any ij

Next consider a magnetic field $\perp$ to v in X

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_x \\ 0 & 0 & -B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu'} = \Lambda^\mu{}_2 \Lambda^\nu{}_3\ F^{23} + \Lambda^\mu{}_3 \Lambda^\nu{}_2\ F^{32}$$

$$F^{23'} = \Lambda^2{}_2 \Lambda^3{}_3\ F^{23} + \Lambda^2{}_3 \Lambda^3{}_2\ F^{32}$$

$$= (\Lambda^2{}_2 \Lambda^3{}_3 - \Lambda^2{}_3 \Lambda^3{}_2)\ F^{23}$$

$$= 1\,\gamma\ F^{23}$$

$$\boxed{B_x' = \gamma B_x}$$

next consider E_y

$$\frac{E_y}{c} = F^{02}$$

$$F'^{02} = \Lambda^0{}_\mu \Lambda^2{}_\nu\ F^{\mu\nu}$$

$$= \underbrace{\Lambda^0{}_3 \Lambda^2{}_2}_{\gamma\beta} \underbrace{F^{32}}_{} + \underbrace{\Lambda^0{}_2 \Lambda^2{}_3}_{0} F^{23}$$

This gives

$$\frac{E_y'}{c} = \gamma\beta B_x$$

$$B_x' = \gamma B_x$$

$$\frac{E_y'}{B_x'} = \frac{c\gamma\beta B_x}{\gamma B_x} = c\gamma$$

$\rightarrow c$ when $v \rightarrow c$

Note that in this case $E \rightarrow 0$ when $\beta \rightarrow 0$

The result is that the Electric and magnetic field are part of the electromagnetic field $F^{\mu\nu}$

current: single particle

$$J^\mu = q\frac{dx^\mu}{d\tau} = q\frac{dx^\mu}{dt}\frac{dt}{d\tau}$$

$$= q\gamma(c, \bar{v})$$

recall that $\frac{dx^\mu}{d\tau}$ transforms like a 4 vector.

① Maxwell's equations have the form

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} F^{\mu\nu} = -\mu_0 J^\nu$$

$$\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} \tilde{F}^{\mu\nu} = 0$$

② These equations have the same form in all inertial coordinate systems if $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ transform like rank 2 antisymmetric tensors under Lorentz transformations

③ These require that the 4 current J^μ transforms like a 4 vector density

$$F'^{\mu\nu} = \sum \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$$

$$\tilde{F}'^{\mu\nu} = \sum \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \tilde{F}^{\alpha\beta}$$

Note these may not look consistent — but the second equation can be expressed directly in terms of F

$$0 = \frac{\partial}{\partial x^\alpha} \eta_{\beta\mu} \eta_{\gamma\nu} F^{\mu\nu} +$$

$$\frac{\partial}{\partial x^\beta} \eta_{\gamma\mu} \eta_{\alpha\nu} F^{\mu\nu} +$$

$$\frac{\partial}{\partial x^\gamma} \eta_{\alpha\mu} \eta_{\beta\nu} F^{\mu\nu}$$