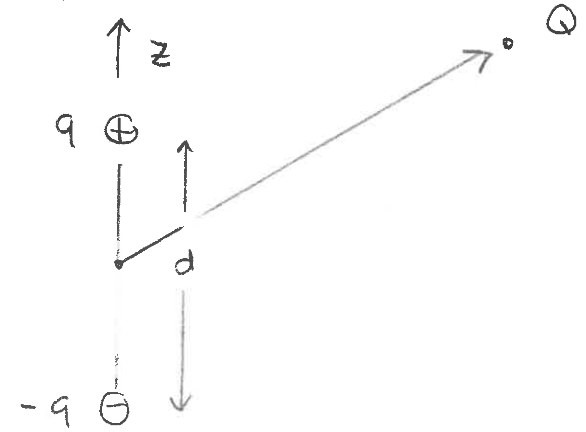
Lecture 3

① Force on charge due to electric dipole

② Continuous charge distribution

③ spherically symmetric charge distribution

④ electric fields



superposition principle + coulomb's law

$$\vec{F}_Q(\vec{r}) = \vec{F}_{Qq_1}(\vec{r}) + \vec{F}_{Q,q_2}(\vec{r})$$

$$= kQq \frac{(x,y,z)-(0,0,\frac{d}{2})}{(x^2+y^2+(z-\frac{d}{2})^2)^{3/2}}$$

$$+ kQ(-q) \frac{(x,y,z)-(0,0,-\frac{d}{2})}{(x^2+y^2+(z+\frac{d}{2})^2)^{3/2}}$$

Last time we used a trick to write this in a different way

$$\frac{x}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{3/2}} = -\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{1/2}}$$

$$\frac{y}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{3/2}} = -\frac{\partial}{\partial y}\frac{1}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{1/2}}$$

$$\frac{z\pm d/2}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{3/2}} = -\frac{\partial}{\partial z}\frac{1}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{1/2}}$$

These are the x, y and z components of

$$\frac{(x,y,z)\pm(0,0,\frac{d}{2})}{(x^2+y^2+(z\pm\frac{d}{2})^2)^{3/2}}$$

This means that we can express $F_Q(\vec{r})$ as

$$\vec{F} = Qkq\underbrace{\left(-\hat{x}\frac{\partial}{\partial x}-\hat{y}\frac{\partial}{\partial y}-\hat{z}\frac{\partial}{\partial z}\right)}_{-\vec{\nabla}}\left(\frac{1}{\sqrt{x^2+y^2+(z-\frac{d}{2})^2}}-\frac{1}{\sqrt{(x^2+y^2+(z+\frac{d}{2})^2)}}\right)$$

$$\vec{\nabla} f(xyz) \equiv \hat{x}\frac{\partial f}{\partial x}+\hat{y}\frac{\partial f}{\partial y}+\hat{z}\frac{\partial f}{\partial z}$$ is called the gradient of $f(\vec{r})$

what can we learn about the dipole if we are very far away? ($|r| >> d$)

$$\frac{1}{\sqrt{x^2+y^2+(z \pm \frac{d}{2})^2}} = \frac{1}{\sqrt{x^2+y^2+z^2 \pm zd + \frac{d^2}{4}}} =$$

let $r^2 = x^2+y^2+z^2$ $\qquad \cos\theta = \frac{z}{r}$

$$\frac{1}{\sqrt{r^2 \pm r^2 \frac{zd}{r^2} + r^2 \frac{d^2}{4r^2}}} = \frac{1}{r} \frac{1}{\sqrt{1 \pm \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}}$$

Forgot

here we assume $\frac{d}{r} << 1$ this means $\left(\frac{d}{r}\right)^2 << \left(\frac{d}{r}\right)$

$$\frac{1}{\sqrt{1+u}} = 1 + \frac{d}{du}\left(\frac{1}{\sqrt{1+u}}\right)\Big|_{u=0} u + O(u^2)$$

$$= 1 - \frac{1}{2}\frac{1}{(\sqrt{1+u})^3}\Big|_{u=0} u + O(u^2)$$

$$= 1 - \frac{1}{2}u + O(u^2)$$

let $u = \pm \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}$

$$\frac{1}{\sqrt{1 \pm \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}} = 1 - \frac{1}{2}\left(\pm \frac{d}{r}\cos\theta\right) \underbrace{- \frac{1}{2}\left(\frac{d^2}{4r^2}\right) + O\left(\left(\frac{d}{r}\right)^2\left(\cos\theta + \frac{d}{4r}\right)^2\right)}_{\text{all much smaller than } d/r}$$

$$\frac{1}{\sqrt{1 \pm \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}} \simeq 1 \mp \frac{1}{2}\frac{d}{r}\cos\theta + o\left(\frac{d^2}{r^2}\right)$$

with this approximation $\bar{F}_Q(\bar{r})$ becomes

$$\bar{F}_Q(\bar{r}) = Qkq\,(-\bar{\nabla})\frac{1}{r}\left(1 + \frac{1}{2}\frac{d}{r}\cos\theta - 1 - \left(-\frac{1}{2}\frac{d}{r}\cos\theta\right) + o\left(\frac{d^2}{r^2}\right)\right)$$

$$= Qkqd\,(-\bar{\nabla})\frac{1}{r}\left(\frac{\cos\theta}{r}\right)$$

$$Qkqd\,(-\bar{\nabla})\frac{1}{r}\left(\frac{z}{r^2}\right)$$

This can be expressed with $\bar{p} = qd\,\hat{z}$

$$\boxed{\bar{F}_Q(\bar{r}) = Q\,(-\bar{\nabla})\left(k\,\frac{\bar{p}\cdot\bar{r}}{r^3}\right) =}$$

we can see that this force depend on the vector $\bar{p}$. (called the dipole moment)

$$= Qkp\,(-\bar{\nabla})\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= Qkp\left(\frac{3xz}{(x^2+y^2+z^2)^{5/2}}, \frac{3yz}{(x^2+y^2+z^2)^{5/2}}, \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\right)$$

$$Qkp\,\frac{3\bar{r}(\bar{r}\cdot\hat{z}) - r^2\hat{z}}{r^5}$$

$$\boxed{F_Q(\vec r) = Q\,k\,\frac{3\vec r(\vec r\cdot\vec p) - r^2\vec p}{r^5}}$$

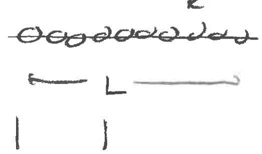
continuous charge distributions

while electric charge comes in multiples of the charge of an electron, for every day phenomena we are dealing with 10^{18} charges.

It is useful for computations to replace them by continuous charge distributions.

<u>line charges</u>

N charges

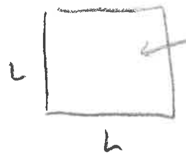


$$\lambda = \frac{Q}{L}$$

λdx = # charges between $| \ |$

$$Q = \int_0^L \lambda\, dx = Q\frac{L}{L} =$$

λ = linear charge density



$$\sigma = \frac{Q}{A}$$

$$A = \text{area} = L^2$$

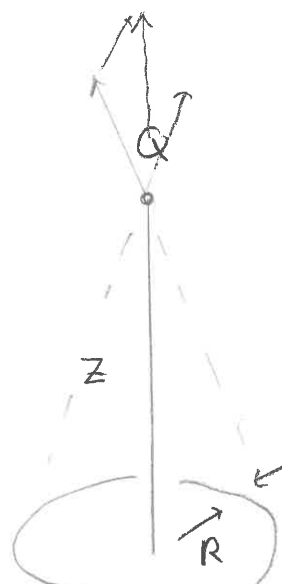
$$Q = \int_{\square} \sigma \, dA = \int_0^L dx \int_0^L dy \; \sigma = L \cdot L \cdot \sigma = Q$$

$\sigma =$ surface charge density

$$\rho = \frac{Q}{V} = \frac{Q}{L^3} = \text{volume charge density}$$

$$Q = \int_V \rho \, dV = \int_0^L dx \int_0^L dy \int_0^L dz \; \rho = L^3 \frac{Q}{L^3}$$

example



ring of positive charge Q_R

$$\lambda = \frac{Q_R}{2\pi R}$$

choose the origin to be at the center of the circle

$$\bar{r} = (0,0,z) \qquad S = R\theta \qquad ds = R\,d\theta$$

$$\bar{r}'(\theta) = (R\cos\theta, R\sin\theta, 0)$$

$$\bar{F}_0(z) = kQ\int_0^{2\pi} (\lambda R\,d\theta)\frac{\bar{r}-\bar{r}'(\theta)}{|\bar{r}-\bar{r}'(\theta)|^3}$$

↑ change in $ds = R\,d\theta$

$$= kQ\int_0^{2\pi} \lambda R\,d\theta \frac{(-R\cos\theta, -R\sin\theta, z)}{|R^2\cos^2\theta + R^2\sin^2\theta + z^2|^{3/2}}$$

$$= kQ\lambda R \frac{R}{(R^2+z^2)^{3/2}}\cdot\Big(\underbrace{-\int_0^{2\pi}\cos\theta\,d\theta}_{(-\sin\theta|_0^{2\pi})\;=\;0}, -\int_0^{2\pi}\sin\theta\,d\theta, \frac{z}{R}\int_0^{2\pi} d\theta\Big)$$

$$\Big(-\sin\theta|_0^{2\pi},\ \cos\theta|_0^{2\pi},\ 2\pi z\Big)$$

$$0 \qquad\qquad 0$$

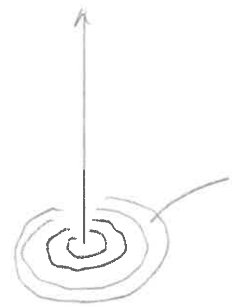
$$= \frac{kQ\lambda \cancel{R^2}\cdot 2\pi z}{(R^2+z^2)^{3/2}}(0,0,1)$$

using $\lambda = \frac{Q_R}{2\pi R}$

$$\boxed{\bar{F}_0(z) = \frac{kQQ_R\cancel{R}z}{(R^2+z^2)^{3/2}}\hat{z}}$$

The net force due to all of the charges on the ring is up as expected.

what if we replace the ring by a disk. We can think of the force due to the disk as the sum of forces due to each ring



$$\sigma = \frac{Q_D}{\pi R^2}$$

$$dQ_R = \sigma \cdot \underbrace{2\pi r\, dr}_{\text{area of ring}}$$

$$\bar{F}_{ring}(z_1) = \frac{kQ\, dQ_R\, r z}{(r^2+z^2)^{3/2}} = \frac{kQ\sigma \cdot 2\pi r\, dr \cdot r z}{(r^2+z^2)^{3/2}}$$

by the superposition principle, the force is the integral from this quantity from $0 \to R$

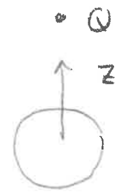
$$\bar{F} = \int_0^R \frac{kQ\sigma \cdot 2\pi r^2\, dr}{(r^2+z^2)^{3/2}} z \cdot \hat{z} =$$

$$= \frac{kQQ_D}{\pi R^2} \cdot 2\pi z \int_0^R \frac{r^2\, dr}{(r^2+z^2)^{3/2}} \hat{z}$$

$$= \frac{2kQQ_D}{R^2} z \int_0^R \frac{r^2\, dr}{(r^2+z^2)^{3/2}} \hat{z}$$

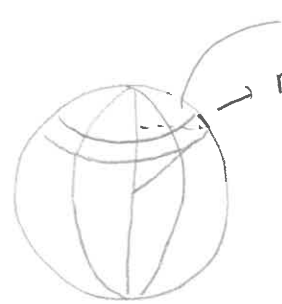
(this integral can be done analytically)

Next we consider a spherical charge distribution $\rho = \rho(r)$



choose the origin of the coordinate system at the center of the spherical charge. Choose the z axis along the line from the center of the charge to Q

spherical volume integrals



$\phi : 0 \to 2\pi$

$\theta : 0 \to \pi$

$$dV = (r\,d\theta)(r\sin\theta\,d\phi)\,dr$$

check $\displaystyle\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \cdot r^2 \sin\theta =$

$$\int_0^R r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$\left(\frac{R^3}{3} - \frac{0^3}{3}\right)\underbrace{(-\cos\pi + \cos 0)}_{2}\,\overbrace{(2\pi - 0)} = \frac{4}{3}\pi R^3$$

which is the volume of a sphere

$$\bar{r} = (0\ 0\ z)$$

$$\bar{r}' = (x\ y\ z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

$$|\bar{r}-\bar{r}'|^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi$$
$$z^2 - 2zr\cos\theta + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos^2\theta + z^2 - 2rz\cos\theta$$

$$= r^2 + z^2 - 2rz\cos\theta$$

$$dQ = \rho\, dV = \rho r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$F_0(z) = kq\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\ \rho(r) r^2 \sin\theta \times$$

$$\frac{(-r\sin\theta\cos\phi, -r\sin\theta\sin\phi, z - r\cos\theta)}{(r^2+z^2-2rz\cos\theta)^{3/2}}$$

This is just the superposition principle. Integrating each component separately

$$\int_0^{2\pi} \cos\phi = \int_0^{2\pi} \sin\phi = 0$$

so the x and y components of the force vanish what remains is in the z direction - there is a 2π from the ϕ integral

$$F_{z0}(z) = kQ\,2\pi \int_0^R \rho(r)\, r^2\, dr \int_0^\pi \frac{z - r\cos\theta}{(r^2+z^2-2rz\cos\theta)^{3/2}} \sin\theta\, d\theta$$

To do the integral use

$$-\frac{d}{dz}\frac{1}{(r^2+z^2-2rz\cos\theta)^{1/2}} = -\left(-\frac{1}{2}\right)\frac{2z-2r\cos\theta}{(r^2+z^2-2rz\cos\theta)^{3/2}}$$

$$= \frac{z-r\cos\theta}{(r^2+z^2-2rz\cos\theta)^{3/2}}$$

using this in the integral

$$F_{z0}(z) = 2\pi kQ \int_0^R \rho(r)\, r^2 \left(-\frac{d}{dz}\right)\int_0^\pi \frac{\sin\theta\, d\theta}{(r^2+z^2-2rz\cos\theta)^{1/2}}$$

let $V = r^2 + z^2 - 2rz\cos\theta$ (replace θ)

$dV = 2rz\sin\theta\, d\theta$ $\quad V: |r-z|^2 \to |r+z|^2$

$\sin\theta\, d\theta = \frac{dV}{2rz}$

$$F_{z0}(z) = 2\pi kQ \int_0^R \rho(r)\, r^2 \left(-\frac{d}{dz}\right)\int_{|z-r|}^{|z+r|^2} \left(\frac{1}{2rz}\right)\frac{dV}{V^{1/2}}$$

$$= 2\pi kQ \int_0^R \rho(r)\, r\left(-\frac{d}{dz}\right)\underbrace{\left(\frac{1}{z}\left(|z+r| - |z-r|\right)\right)}_{z>r \;\to\; 2\frac{r}{z^2}}$$

$z<r \quad \frac{d}{dz}(2) = 0$

case 1 $z > R$

$$F = \frac{4\pi k Q}{z^2} \int \rho(r) r^2 dr = \frac{k Q Q_{sphere}}{z^2}$$

case 2 $z = r < R$

$$F = \frac{4\pi k Q}{z^2} \int_0^z \rho(r) r^2 dr = \frac{k Q}{z^2}$$ charge inside sphere of radius z

conclusions

* spherical charge distribution gives the same force as a point charge at the origin with the same charge

(note z is just the distance between the particle and the center of the sphere

* there are no forces on a charge inside of a hollow sphere

electric fields

In coulombs law

$$F_Q(r) = \sum kQq_i \frac{\bar{r}-\bar{r}_i}{|r-\bar{r}_i|^3} = Q\left(k\sum q_i \frac{\bar{r}-\bar{r}_i}{|r-r_i|^3}\right)$$

the quantity in ( ) only depends on $\bar{r}$ and the other charges. It is a vector called the electric field at $\bar{r}$

The above equation can be expressed as

$$\bar{F}_Q(r) = Q\bar{E}(\bar{r})$$

$$\bar{E}(r) = \frac{\bar{F}_Q(r)}{Q}$$

is called the <u>electric</u> <u>field</u> at $\bar{r}$

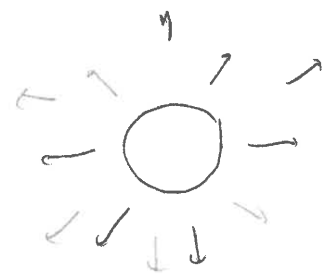
* It exists in the absence of the charge Q

* To measure the electric field at $\bar{r}$

$$\bar{E}(r) = \lim_{Q\to 0} \frac{\bar{F}_Q(r)}{Q}$$

this prevents Q from distrubing the position of the other charges

$\bar{E}(\bar{r})$ assigns a vector at each point is space. The direction is the direction that a positive charge would move if placed at that point



Q = charge

hollow sphere - field outside is radical, no field inside